CHAPTER 4

Complex Evolutionary Systems in Behavioral Finance

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Abstract

Traditional finance is built on the rationality paradigm. This chapter discusses simple models from an alternative approach in which financial markets are viewed as complex evolutionary systems. Agents are boundedly rational and base their investment decisions on market-forecasting heuristics. Prices and beliefs about future prices co-evolve over time with mutual feedback. Strategy choice is driven by evolutionary selection so that agents tend to adopt strategies that were successful in the past. Calibration of “simple complexity models” with heterogeneous expectations to real financial market data and laboratory experiments with human subjects are also discussed.

Keywords: stylized facts, power laws, agent-based models, interacting agents

4.1. INTRODUCTION

Finance is witnessing important changes, according to some even a paradigmatic shift, from the traditional, neoclassical mathematical modeling approach based on a representative, fully rational agent and perfectly efficient markets (Muth, 1961; Lucas, 1971; Fama, 1970) to a behavioral approach based on computational models where markets are viewed as complex evolving systems with many interacting, “boundedly rational” agents using simple “rule of thumb” trading strategies (e.g., Anderson et al., 1988; Brock, 1993; Arthur, 1995; Arthur et al., 1997a; Tesfatsion and Judd, 2006). Investor’s psychology plays a key role in behavioral finance, and different types of psychology-based trading and behavioral modes have been identified in the literature, such as positive feedback or momentum trading, trend extrapolation, noise trading, overconfidence, overreaction, optimistic or pessimistic traders, upward- or downward-biased traders, correlated imperfect rational trades, overshooting, contrarian strategies, and so on. Some key references dealing with various aspects of investor psychology include, for example, Cutler et al. (1989), DeBondt and Thaler (1985), DeLong et al. (1990a,b), Brock and Hommes (1997, 1998), Gervais and Odean (2001), and Hong and Stein (1999, 2003), among others—see, for example, Shleifer (2000), Hirschleifer (2001), and Barberis and Thaler (2003) for extensive surveys and many more references on behavioral finance.

An important problem of a behavioral approach is that it leaves “many degrees of freedom.” There are many ways individual agents can deviate from full rationality. Evolutionary selection based on relative performance is one plausible way to discipline the “wilderness of bounded rationality.” Milton Friedman (1953) argued that
nonrational agents will not survive evolutionary competition and will therefore be driven out of the market, thus providing support to a representative rational agent framework as a (long run) description of the economy. In the same spirit, Alchian (1950) argued that biological evolution and natural selection driven by realized profits may eliminate nonrational, nonoptimizing firms and lead to a market in which rational, profit maximizing firms dominate. Blume and Easley (1992, 2006) have, however, shown that the market selection hypothesis does not always hold and that nonrational agents may survive in the market. Brock (1993, 1997), Arthur et al. (1997b), LeBaron et al. (1999), and Farmer (2002), among others, introduced artificial stock markets, described by agent-based models with evolutionary selection between many different interacting trading strategies. They showed that the market does not generally select for the rational, fundamental strategy and that simple technical trading strategies may survive in artificial markets. Computationally oriented agent-based simulation models have been reviewed in LeBaron (2006); see also the special issue of the Journal of Mathematical Economics (Hens and Schenk-Hoppé, 2005) and the survey chapter of Evstigneev, Hens, and Schenk-Hoppé (2009) in this book for an overview of evolutionary finance.1

Stimulated by work on artificial markets, in the last decade quite a number of “simple complexity models” have been introduced. Markets are viewed as evolutionary adaptive systems with boundedly rational interacting agents, but the models are simple enough to be at least partly analytically tractable. The study of simple complexity models typically requires a well-balanced mixture of analytical and computational tools. This literature is surveyed in Hommes (2006) and Chiarella et al. (2009); see also Lux (2009), who discusses in detail how well models with interacting agents match important stylized facts such as fat tails in the returns distribution and long memory. Without repeating an extensive survey, this chapter focuses on a number of simple examples, in particular the adaptive belief systems (ABSs) of Brock and Hommes (1997, 1998). These models serve as didactic examples of nonlinear dynamic asset-pricing models with evolutionary strategy switching, and they illustrate some of the key features present in the interacting agents literature. The model also has been used to test the relevance of the theory of heterogeneous expectations empirically as well as in laboratory experiments with human subjects. Simple complexity models may also be used by practitioners or policy makers. To illustrate this point, we present an example showing how such a model can be used to evaluate how likely it is that a stock market bubble will resume.

Two important features of the ABS are that agents are boundedly rational and that they have heterogeneous expectations. An ABS is in fact a standard discounted value asset-pricing model derived from mean-variance maximization, extended to the case of heterogeneous beliefs. Two classes of investors that are also observed in financial practice can be distinguished: fundamentalists and technical analysts. Fundamentalists base their forecasts of future prices and returns on economic fundamentals, such as dividends, interest rates, price-earning ratios, and so on. In contrast, technical analysts are looking for patterns in past prices and base their forecasts upon extrapolation of

1Some other recent references are Amir et al. (2005) and Evstigneev et al. (2002, 2008).
these patterns. Fractions of these two types of traders are time varying and depend on relative performance. Strategy choice is thus based on evolutionary selection or reinforcement learning, with agents switching to more successful (i.e., profitable) rules. Asset price fluctuations are characterized by irregular switching between a stable phase when fundamentalists dominate the market and an unstable phase when trend followers dominate and asset prices deviate from benchmark fundamentals. Price deviations from the rational expectations fundamental benchmark and excess volatility are triggered by news about economic fundamentals but may be \textit{amplified} by evolutionary selection, based on recent performance, of trend-following strategies.

There is empirical evidence that experience-based reinforcement learning plays an important role in investment decisions in real markets. For example, Ippolito (1992), Chevalier and Ellison (1997), Sirri and Tufano (1998), Rockinger (1996), and Karceski (2002) show for mutual funds data that money flows into past good performers while flowing out of past poor performers and that performance persists on a short-term basis. Pension funds are less extreme in picking good performance but are tougher on bad performers (Del Guercio and Tkac, 2002). Benartzi and Thaler (2007) have shown that heuristics and biases play a significant role in retirement savings decisions. For example, using data from Vanguard, they show that the equity allocation of new participants rose from 58% in 1992 to 74% in 2000, following a strong rise in stock prices in the late 1990s; however, it dropped back to 54% in 2002, following the extreme fall in stock prices.

Laboratory experiments with human subjects have shown that individuals often do not behave fully rationally but tend to use heuristics, possibly biased, in making economic decisions under uncertainty (Kahneman and Tversky, 1973). In a similar vein, Smith et al. (1998) have shown the occurrence of bubbles and the ease with which markets deviate from full rationality in asset-pricing laboratory experiments. These bubbles occur despite the fact that participants had sufficient information to compute the fundamental value of the asset. Laboratory experiments with human subjects provide an important tool to investigate which behavioral rules play a significant role in deviations from the rational benchmark, and they can thus help discipline the class of behavioral modes. Duffy (2008) gives a stimulating recent overview concerning the role of laboratory experiments to explain macro phenomena.

Heterogeneity in forecasting future asset prices is supported by evidence from survey data. For example, Vissing-Jorgensen (2003) reports that at the beginning of 2000, 50% of individual investors considered the stock market to be overvalued, approximately 25% believed that it was fairly valued, about 15% were unsure, and less than 10% believed that it was undervalued. This is an indication of heterogeneous beliefs among individual investors about the prospect of the stock market. Similarly, Shiller (2000) finds evidence that investors’ sentiment varies over time. Both institutional and individual investors become more optimistic in response to significant increases in the recent performance of the stock market.

This chapter is organized as follows. Section 4.2 introduces the main features of adaptive belief systems, and Section 4.3 discusses a number of simple examples with two, three, and four different trader types. In Section 4.4 an analytical framework with
many different trader types is presented. Section 4.5 discusses the empirical relevance of behavioral heterogeneity. The estimation of a simple model with fundamentalists and chartists on yearly S&P 500 data shows how the worldwide stock market bubble in the late 1990s, triggered by good news about fundamentals (a new Internet technology), may have been strongly amplified by trend-following strategies. Section 4.6 reviews some learning to forecast laboratory experiments with human subjects, investigating which individual forecasting rules agents may use, how these rules interact, and which aggregate outcome they cocreate. Section 4.7 concludes the chapter, sketching some challenges for future research and potential applications for financial practitioners and policy makers. An appendix contains a short mathematical overview of bifurcation theory, which plays a role in the transition to complicated price fluctuations in the simple complexity models discussed in this chapter.

### 4.2. AN ASSET-PRICING MODEL WITH HETEROGENEOUS BELIEFS

This section discusses the asset-pricing model with heterogeneous beliefs as introduced in Brock and Hommes (1998), using evolutionary selection of expectations as in Brock and Hommes (1997a). This simple modeling framework has been inspired by computational work at the Santa Fe Institute (SFI) and may be viewed as a simple, partly analytically tractable version of the more complicated SFI artificial stock market of Arthur et al. (1997b).

Agents can invest in either a risk-free asset or a risky asset. The risk-free asset is in perfect elastic supply and pays a fixed rate of return $r$; the risky asset pays an uncertain dividend. Let $p_t$ be the price per share (ex-dividend) of the risky asset at time $t$, and let $y_t$ be the stochastic dividend process of the risky asset. Wealth dynamics is given by

$$W_{t+1} = RW_t + (p_{t+1} + y_{t+1} - Rp_t)z_t$$ (4.1)

where $R = 1 + r$ is the gross rate of risk-free return and $z_t$ denotes the number of shares of the risky asset purchased at date $t$. Let $E_{ht}$ and $V_{ht}$ denote the “beliefs” or forecasts of trader type $h$ about conditional expectation and conditional variance.

Agents are assumed to be myopic mean-variance maximizers so that the demand $z_{ht}$ of type $h$ for the risky asset solves

$$\text{Max}_{z_t} \{E_{ht}[W_{t+1}] - \frac{a}{2}V_{ht}[W_{t+1}]\}$$ (4.2)

where $a$ is the risk-aversion parameter. The demand $z_{ht}$ for risky assets by trader type $h$ is then

$$z_{ht} = \frac{E_{ht}[p_{t+1} + y_{t+1} - Rp_t]}{\alpha V_{ht}[p_{t+1} + y_{t+1} - Rp_t]} = \frac{E_{ht}[p_{t+1} + y_{t+1} - Rp_t]}{a\sigma^2}$$ (4.3)
where the conditional variance \( V_{ht} = \sigma^2 \) is assumed to be constant and equal for all types.\(^2\) Let \( z^s \) denote the supply of outside risky shares per investor, also assumed to be constant, and let \( n_{ht} \) denote the fraction of type \( h \) at date \( t \). Equilibrium of demand and supply yields

\[
\sum_{h=1}^{H} n_{ht} \frac{E_{ht}[p_{t+1} + y_{t+1} - R p_t]}{a \sigma^2} = z^s
\]

(4.4)

where \( H \) is the number of different trader types.

The forecasts \( E_{ht}[p_{t+1} + y_{t+1}] \) of tomorrow’s prices and dividends are made before the equilibrium price \( p_t \) has been revealed by the market and therefore will depend on a publically available information set \( I_{t-1} = \{ p_{t-1}, p_{t-2}, \ldots ; y_{t-1}, y_{t-2}, \ldots \} \) of past prices and dividends. Solving the heterogeneous market-clearing equation for the equilibrium price gives

\[
R p_t = \sum_{h=1}^{H} n_{ht} E_{ht}[p_{t+1} + y_{t+1}] - a \sigma^2 z^s
\]

(4.5)

The quantity \( a \sigma^2 z^s \) may be interpreted as a risk premium for traders to hold risky assets.

### 4.2.1. The Fundamental Benchmark with Rational Agents

When all agents are identical and expectations are homogeneous, the equilibrium pricing Eq. 4.5 reduces to

\[
R p_t = E_t[p_{t+1} + y_{t+1}] - a \sigma^2 z^s
\]

(4.6)

where \( E_t \) is the common conditional expectation in the beginning of period \( t \). It is well known that, assuming that a transversality condition \( \lim_{t \to \infty} \frac{E_t[p_{t+k}]}{R^k} = 0 \) holds, the price of the risky asset is given by the discounted sum of expected future dividends minus the risk premium:

\[
p_t^* = \sum_{k=1}^{\infty} \frac{E_t[y_{t+k}]}{R^k} - a \sigma^2 z^s
\]

(4.7)

The price \( p_t^* \) in Eq. 4.7 is called the fundamental rational expectations price, or the fundamental price for short. It is completely determined by economic fundamentals, which are here given by the stochastic dividend process \( y_t \). In this section we focus on the case of an independently identically distributed (IID) dividend process \( y_t \), but the estimation of the simple two-type model discussed in Section 4.5 uses a nonstationary dividend process.\(^3\) For the special case of an IID dividend process \( y_t \), with constant

\(^2\)Gaunersdorfer (2000) investigates the case with time-varying beliefs about variances and shows that the asset price dynamics are quite similar. Chiarella and He (2002, 2003) investigate the model with heterogeneous risk-aversion coefficients.

\(^3\)Brock and Hommes (1997b) also discuss a nonstationary example, in which the dividend process follows a geometric random walk.
mean $E[y_t] = \bar{y}$, the fundamental price is constant:

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y} - a\sigma^2 z^s}{R^k} = \frac{\bar{y} - a\sigma^2 z^s}{r} \quad (4.8)$$

Recall that, in addition to the rational expectations fundamental solution in Eq. 4.7, so-called rational bubble solutions of the form $p_t = p^*_t + (1 + r)^t(0 - p^*_0)$ also satisfy the pricing Eq. 4.6. Along these bubble solutions, traders have rational expectations (perfect foresight), but they are ruled out by the transversality condition. In a perfectly rational world, traders realize that such bubbles cannot last forever, and therefore all traders believe that the value of a risky asset is always equal to its fundamental price. Changes in asset prices are then only driven by unexpected changes in dividends and random “news” about economic fundamentals. In a heterogeneous world the situation will be quite different.

### 4.2.2. Heterogeneous Beliefs

It will be convenient to work with the deviation from the fundamental price

$$x_t = p_t - p^*_t \quad (4.9)$$

We make the following assumptions about the beliefs of trader type $h$:

- **B1** $V_{ht}[p_{t+1} + y_{t+1} - Rp_t] = V_t[p_{t+1} + y_{t+1} - Rp_t] = \sigma^2$, for all $h, t$.
- **B2** $E_{ht}[y_{t+1}] = E_t[y_{t+1}] = \bar{y}$, for all $h, t$.
- **B3** All beliefs $E_{ht}[p_{t+1}]$ are of the form

$$E_{ht}[p_{t+1}] = E_t[p^*_{t+1}] + E_{ht}[x_{t+1}] = p^* + f_h(x_{t-1}, \ldots, x_{t-L}) \quad (4.10)$$

for all $h, t$.

According to B1, beliefs about conditional variance are equal and constant for all types, as discussed already. Assumption B2 states that all types have correct expectations about future dividends $y_{t+1}$ given by the conditional expectation, which is $\bar{y}$ in the case of IID dividends. According to B3, beliefs about future prices consist of two parts: a common belief about the fundamental plus a heterogeneous part $f_h$. Each forecasting rule $f_h$ represents a model of the market (e.g., a technical trading rule) according to which type $h$ believes that prices will deviate from the fundamental price.

An important and convenient consequence of the assumptions B1–B3 about traders’ beliefs is that the heterogeneous agent market equilibrium Eq. 4.5 can be reformulated in deviations from the benchmark fundamental. In particular, substituting the price forecast

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4The assumption that all types know the fundamental price is without loss of generality, because any forecasting rule not using the fundamental price can be reparameterized or reformulated for mathematical convenience in deviations from an (unknown) fundamental price $p^*$. 
(Eq. 4.10) in the market equilibrium Eq. 4.5 and using \( R^*_t = E_t[p^*_t + y_{t+1}] - a \sigma^2 z^s \) yields the equilibrium equation in deviations from the fundamental:

\[
R_t = \sum_{h=1}^{H} n_{ht} E_{ht}[x_{t+1}] \equiv \sum_{h=1}^{H} n_{ht} f_{ht} \tag{4.11}
\]

with \( f_{ht} = f_h(x_{t-1}, \ldots, x_{t-L}) \). Note that the benchmark fundamental is nested as a special case within this general setup, with all forecasting strategies \( f_h \equiv 0 \). Hence, the adaptive belief systems can be used in empirical and experimental testing where asset prices deviate significantly from some benchmark fundamental.

### 4.2.3. Evolutionary Dynamics

The evolutionary part of the model describes how beliefs are updated over time, that is, how the fractions \( n_{ht} \) of trader types evolve over time. These fractions are updated according to an evolutionary fitness or performance measure. The fitness measures of all trading strategies are publically available but subject to noise. Fitness is derived from a random utility model and given by

\[
\bar{U}_{ht} = U_{ht} + \varepsilon_{iht} \tag{4.12}
\]

where \( U_{ht} \) is the deterministic part of the fitness measure and \( \varepsilon_{iht} \) represents an individual agent’s IID error when perceiving the fitness of strategy \( h = 1, \ldots H \).

To obtain analytical expressions for the probabilities or fractions, the noise term \( \varepsilon_{iht} \) is assumed to be drawn from a double exponential distribution. As the number of agents goes to infinity, the probability that an agent chooses strategy \( h \) is given by the multinomial logit model (or “Gibbs” probabilities):\(^5\)

\[
n_{ht} = \frac{e^{\beta U_{ht-1}}}{\sum_{h=1}^{H} e^{\beta U_{ht-1}}} \tag{4.13}
\]

Note that the fractions \( n_{ht} \) add up to 1.

A key feature of Eq. 4.13 is that the higher the fitness of trading strategy \( h \), the more traders will select strategy \( h \). Hence, Eq. 4.13 represents a form of reinforcement learning: Agents tend to switch to strategies that have performed well in the (recent) past. The parameter \( \beta \) in Eq. 4.13 is called the intensity of choice; it measures the sensitivity of the mass of traders to selecting the optimal prediction strategy. The intensity of choice \( \beta \) is inversely related to the variance of the noise terms \( \varepsilon_{iht} \). The extreme case \( \beta = 0 \) corresponds to noise of infinite variance so that differences in fitness cannot be observed and all fractions (4.13) will be fixed over time and equal to \( 1/H \).

The other extreme case \( \beta = +\infty \) corresponds to the case without noise so that the deterministic part of the fitness can be observed perfectly, and in each period, all traders

choose the optimal forecast. An increase in the intensity of choice $\beta$ represents an increase in the degree of rationality with respect to evolutionary selection of trading strategies. The timing of the coupling between the market equilibrium Eq. 4.5 or 4.11 and the evolutionary selection of strategies (Eq. 4.13) is important. The market equilibrium price $p_t$ in Eq. 4.5 depends on the fractions $n_{ht}$. The notation in Eq. 4.13 stresses the fact that these fractions $n_{ht}$ depend on most recently observed past fitnesses $U_{h,t-1}$, which in turn depend on past prices $p_{t-1}$ and dividends $y_{t-1}$ in periods $t-1$ further in the past, as shown next. After the equilibrium price $p_t$ has been revealed by the market, it will be used in evolutionary updating of beliefs and determining the new fractions $n_{h,t+1}$. These new fractions will then determine a new equilibrium price $p_{t+1}$, and so on. In an adaptive belief system, market equilibrium prices and fractions of different trading strategies thus coevolve over time.

A natural candidate for evolutionary fitness is (a weighted average of) realized profits, given by

$$U_{ht} = (p_t + y_t - R p_{t-1}) \frac{E_{h,t-1}[p_t + y_t - R p_{t-1}]}{a \sigma^2} + w U_{h,t-1} \tag{4.14}$$

where $0 \leq w \leq 1$ is a memory parameter measuring how fast past realized fitness is discounted for strategy selection.

Fitness can be rewritten in terms of deviations from the fundamental as

$$U_{ht} = (x_t - Rx_{t-1} + a \sigma^2 z + \delta_t) \left( \frac{f_{h,t-1} - Rx_{t-1} + a \sigma^2 z}{a \sigma^2} \right) + w U_{h,t-1} \tag{4.15}$$

where $\delta_t \equiv p^*_t + y_t - E_{t-1}[p^*_t + y_t]$ is a martingale difference sequence.

### 4.2.4. Forecasting Rules

To complete the model, we have to specify the class of forecasting rules. Brock and Hommes (1998) have investigated evolutionary competition between simple linear forecasting rules with only one lag, that is,

$$f_{ht} = g_h x_{t-1} + b_h \tag{4.16}$$

It can be argued that, for a forecasting rule to have any impact in real markets, it has to be simple, because it seems unlikely that enough traders will coordinate on a

---

$^6$Note that this fitness measure does not take into account the risk taken at the moment of the investment decision. In fact, one could argue that the fitness measure (Eq. 4.14) does not take into account the variance term in (Eq. 4.2) capturing the investors’ risk taken before obtaining that profit. On the other hand, in real markets, realized net profits or accumulated wealth may be what investors care about most, and the non-risk adjusted fitness measure (Eq. 4.14) may thus be of relevance in practice. See also DeLong et al. (1990) for a discussion of this point. Given that investors are risk averse, mean-variance maximizers maximizing their expected utility from wealth (Eq. 4.2), an alternative natural candidate for fitness, the risk-adjusted profit given by $\pi_{ht} = R_h z_{h,t-1} - \frac{1}{2} \sigma^2 z^2_{h,t-1}$, where $R_t = p_t + y_t - R p_{t-1}$ and $z_{h,t-1} = E_{h,t-1}[R_t] / (a \sigma^2)$ is the demand by trader type $h$. Hommes (2001) shows that the risk-adjusted fitness measure is, up to a type-independent level, equivalent to minus-squared prediction errors.
complicated rule. The simple linear rule (Eq. 4.16) includes a number of important special cases. For example, when both the trend and the bias parameters \( g_h = b_h = 0 \), the rule reduces to the \textit{fundamentalists} forecast, that is,

\[
f_{ht} \equiv 0
\]  

(4.17)

predicting that the deviation \( x \) from the fundamental will be 0, or equivalently, that the price will be at its fundamental value. Other important cases covered by the linear forecasting rule (Eq. 4.16) are the pure \textit{trend followers}:

\[
f_{ht} = g_h x_{t-1} \quad g_h > 0
\]  

(4.18)

and the pure \textit{biased belief}

\[
f_{ht} = b_h
\]  

(4.19)

Notice that the simple pure bias forecast (Eq. 4.19) represents any positively or negatively biased forecast of next period’s price that traders might have. Instead of these extremely simple habitual rule-of-thumb forecasting rules, some might prefer the rational, \textit{perfect foresight} forecasting rule:

\[
f_{ht} = x_{t+1}
\]  

(4.20)

We emphasize, however, that the perfect foresight forecasting rule (Eq. 4.20) assumes perfect knowledge of the heterogeneous market equilibrium Eq. 4.5, and in particular perfect knowledge about the beliefs of all other traders. Although the case with perfect foresight has much theoretical appeal, its practical relevance in a complex heterogeneous world should not be overstated, since this underlying assumption seems rather strong.\(^7\)

\section*{4.3. SIMPLE EXAMPLES}

This section presents simple but typical examples of \textit{adaptive belief systems} (ABSs), with two, three, or four competing \textit{linear} forecasting rules (Eq. 4.16), where the parameter \( g_h \) represents a perceived \textit{trend} in prices and the parameter \( b_h \) represents a perceived upward or downward \textit{bias}. The ABS with \( H \) types is given by (in deviations

\(^7\)Brock and Hommes (1997) analyze the cobweb model with costly rational versus cheap naive expectations and find irregular price fluctuations due to endogenous switching between free riding and costly rational forecasting. In general, however, a \textit{temporary equilibrium} model with heterogeneous beliefs, such as the asset-pricing model, is difficult to analyze if one of the types has perfect foresight. Brock et al. (2008) discuss how a perfect foresight trader may affect the dynamics in an asset-pricing model with heterogeneous beliefs.
from the fundamental benchmark):

\[(1 + r)x_t = \sum_{h=1}^{H} n_{ht}(ghx_{t-1} + b_h) + \epsilon_t \tag{4.21}\]

\[n_{ht} = \frac{e^{\beta U_{h,t-1}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}} \tag{4.22}\]

\[U_{h,t-1} = (x_{t-1} - Rx_{t-2}) \left( \frac{ghx_{t-3} + b_h - Rx_{t-2}}{a\sigma^2} \right) + wU_{h,t-2} - C_h \tag{4.23}\]

where \(\epsilon_t\) is a small noise term representing, for example, a small fraction of noise traders and/or random outside supply of the risky asset.

To keep the analysis of the dynamical behavior tractable, Brock and Hommes (1998) focused on the case where the memory parameter \(w = 0\), so that evolutionary fitness is given by last period’s realized profit. A common feature of all examples is that, as the intensity of choice to switch prediction or trading strategies increases, the fundamental steady state becomes locally unstable and nonfundamental steady states, cycles, or even chaos arise. In the examples that follow, we encounter different bifurcation routes (i.e., transitions) to complicated dynamics. A mathematical appendix summarizes the most important bifurcations, that is, qualitative changes in the dynamics (e.g., when a steady state loses stability or a new cycle is created) when a model parameter changes.

### 4.3.1. Costly Fundamentalists vs. Trend Followers

The simplest example of an ABS only has two trader types, with forecasting rules

\[f_{1t} = 0 \quad \text{fundamentalists} \tag{4.24}\]

\[f_{2t} = gx_{t-1} \quad g > 0 \quad \text{trend followers} \tag{4.25}\]

The first type are fundamentalists predicting that the price will equal its fundamental value (or equivalently that the deviation will be zero) and the second type are pure trend followers predicting that prices will rise (or fall) by a constant rate. In this example, the fundamentalists have to pay a fixed per-period positive cost \(C_1\) for information gathering; in all other examples discussed later, information costs are set to zero for all trader types.

For small values of the trend parameter \(0 \leq g < 1 + r\), the fundamental steady state is always stable. Only for sufficiently high trend parameters, \(g > 1 + r\), trend followers can destabilize the system. For trend parameters \(1 + r < g < (1 + r)^2\), the dynamic behavior of the evolutionary system depends on the intensity of choice to switch between the two trading strategies.\(^8\) For low values of the intensity of choice,

\(^8\)For \(g > (1 + r)^2\) the system may become globally unstable and prices may diverge to infinity. Imposing a stabilizing force—for example, by assuming that trend followers condition their rule on deviations from the fundamental, as in Gaunersdorfer, Hommes, and Wagener (2008)—leads to a bounded system again, possibly with cycles or even chaotic fluctuations.
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FIGURE 4.1  Time series of price deviations from fundamental (a) and fractions of fundamentalists (b) and attractor (c) in the \((x_t, n_{t1})\)-phase space for two-type model with costly fundamentalist versus trend followers buffeted with small noise (SD = 0.1). Price dynamics are characterized by temporary bubbles when trend followers dominate the market, interrupted by sudden crashes, when fundamentalists dominate. In the presence of (small) noise, the system switches back and forth between two coexisting quasiperiodic attractors of the underlying deterministic skeleton, one with prices above and one with prices below its fundamental value. Parameters are \(\beta = 3.6\), \(g = 1.2\), \(R = 1.1\), and \(C = 1\).

The fundamental steady state will be stable. As the intensity of choice increases, the fundamental steady state becomes unstable due to a pitchfork bifurcation in which two additional nonfundamental steady states \(-x^* < 0 < x^*\) are created.

As the intensity of choice increases further, the two nonfundamental steady states also become unstable due to a Hopf bifurcation, and limit cycles or even strange attractors can arise around each of the (unstable) nonfundamental steady states.\(^9\) The evolutionary ABS may cycle around the positive nonfundamental steady state, cycle around the negative nonfundamental steady state, or, driven by the noise, switch back and forth between cycles around the high and the low steady states, as illustrated in Figure 4.1. The simulations use the E&F Chaos software as discussed in Diks et al. (2008).

This example shows that, in the presence of information costs and with zero memory, when the intensity of choice in evolutionary switching is high, fundamentalists cannot

\(^9\)See Appendix 4.2 for a more detailed discussion of the pitchfork bifurcation and the Hopf bifurcation.
FIGURE 4.2  Bifurcation diagram (a) and largest Lyapunov exponent plot (b) as a function of the intensity of choice $\beta$ for two-type model with costly fundamentalist versus trend followers. In both plots the model is buffeted with very small noise ($SD = 10^{-6}$) for the noise term $\epsilon_t$ in Eq. 4.21; to avoid that for large $\beta$ values the system gets stuck in the locally unstable steady state. Parameters are $g = 1.2$, $R = 1.1$, $C = 1$, and $2 \leq \beta \leq 4$. A pitchfork bifurcation of the fundamental steady state, in which two stable nonfundamental steady states are created, occurs for $\beta \approx 2.37$. The nonfundamental steady states become unstable due to a Hopf bifurcation for $\beta \approx 3.33$, and (quasi-)periodic dynamics arise. For large values of $\beta$ the largest Lyapunov exponent becomes positive, indicating chaotic price dynamics.

drive out pure trend followers and persistent deviations from the fundamental price may occur.\textsuperscript{10}

Figure 4.2 illustrates that the asset-pricing model with costly fundamentalists versus cheap trend-following exhibits a rational route to randomness, that is, a bifurcation route to chaos occurs as the intensity of choice to switch strategies increases.

4.3.2. Fundamentalists vs. Opposite Biases

The second example of an ABS is an example with three trader types without any information costs. The forecasting rules are

$$
\begin{align*}
  f_{1t} &= 0 \quad \text{fundamentalists} \\
  f_{2t} &= b \quad b > 0, \quad \text{positive bias (optimists)} \\
  f_{3t} &= -b \quad -b < 0 \quad \text{negative bias (pessimists)}
\end{align*}
$$

The first type are fundamentalists as before, but there are no information costs for fundamentalists. The second and third types have a purely biased belief, expecting a constant price above and below, respectively, the fundamental price.

For low values of the intensity of choice, the fundamental steady state is stable. As the intensity of choice increases the fundamental steady becomes unstable due to a Hopf bifurcation and the dynamics of the ABS is characterized by cycles around the

\textsuperscript{10}Brock and Hommes (1999) show that this result also holds when the memory in the fitness measure increases. In fact, an increase in the memory of the evolutionary fitness leads to bifurcation routes very similar to bifurcation routes that are due to an increase in the intensity of choice.
unstable steady state. This example shows that, even when there are \textit{no} information costs for fundamentalists, they cannot drive out other trader types with opposite biased beliefs. In the evolutionary ABS with high intensity of choice, fundamentalists and biased traders coexist with fractions varying over time and prices fluctuating around the unstable fundamental steady state.

Moreover, Brock and Hommes (1998, p. 1259, Lemma 9) show that as the intensity of choice tends to infinity the ABS converges to a (globally) stable cycle of Period 4. Average profits along this four-cycle are equal for all three trader types. Hence, if the initial wealth is equal for all three types, in this evolutionary system in the long run, accumulated wealth will be equal for all three types. This example shows that the Friedman argument that smart fundamental traders will always drive out simple rule-of-thumb speculative traders is in general not valid.\footnote{This result is related to DeLong et al. (1990a,b), who show that a constant fraction of noise traders can survive in the market in the presence of fully rational traders. The ABS, however, are evolutionary models with \textit{time-varying} fractions, driven by strategy performance.}

### 4.3.3. Fundamentalists vs. Trend and Bias

The third example of an ABS is an example with \textit{four} trader types, with linear forecasting rules (Eq. 4.16) with parameters $g_1 = 0$, $b_1 = 0$; $g_2 = 0.9$, $b_2 = 0.2$; $g_3 = 0.9$, $b_3 = -0.2$; and $g_4 = 1 + r = 1.01$, $b_4 = 0$. The first type are fundamentalists again, without information costs, and the other three types follow a simple linear forecasting rule with one lag. The dynamical behavior is illustrated in Figures 4.3 and 4.4.

For low values of the intensity of choice, the fundamental steady state is stable. As the intensity of choice increases, as in the previous three-type example, the fundamental steady becomes unstable due to a Hopf bifurcation and a stable invariant circle around the unstable fundamental steady state arises, with periodic or quasi-periodic fluctuations. As the intensity of choice further increases, the invariant circle breaks into a strange attractor with chaotic fluctuations. In the evolutionary ABS, fundamentalists and chartists coexist with time-varying fractions and prices moving chaotically around the unstable fundamental steady state. Figure 4.4 shows that in this four-type example with fundamentalists versus trend followers and biased beliefs a rational route to randomness occurs, with positive largest Lyapunov exponents for large values of $\beta$.

This four-type example shows that, even when there are \textit{no} information costs for fundamentalists, they cannot drive out other simple trader types and fail to stabilize price fluctuations toward its fundamental value. As in the three-type case, the opposite biases create cyclic behavior and trend extrapolation turns these cycles into unpredictable chaotic fluctuations.

### 4.3.4. Efficiency

What can be said about market efficiency in an ABS? The (noisy) chaotic price fluctuations are characterized by an irregular switching between phases of close-to-the-fundamental-price fluctuations, phases of “optimism” with prices following an upward
FIGURE 4.3  Chaotic (a) and noisy chaotic (b) time series of asset prices in an adaptive belief system with four trader types. Strange attractor (c) and enlargement of strange attractor (d). Belief parameters are $g_1 = 0, b_1 = 0; g_2 = 0.9, b_2 = 0.2; g_3 = 0.9, b_3 = -0.2$; and $g_4 = 1 + r = 1.01, b_4 = 0$. Other parameters are $r = 0.01, \beta = 90.5, w = 0$, and $C_h = 0$ for all $1 \leq h \leq 4$.

FIGURE 4.4  Bifurcation diagram (a) and largest Lyapunov exponent plot (b) for the four-type model, buffeted with very small noise ($SD = 10^{-6}$ for noise term $\epsilon_t$ in Eq. 4.21) to avoid that for large $\beta$-values the system gets stuck in the locally unstable steady state. Belief parameters are $g_1 = 0, b_1 = 0; g_2 = 0.9, b_2 = 0.2; g_3 = 0.9, b_3 = -0.2$; and $g_4 = 1 + r = 1.01, b_4 = 0$. Other parameters are $r = 0.01, \beta = 90.5, w = 0$, and $C_h = 0$ for all $1 \leq h \leq 4$. The four-type model with fundamentalists versus trend followers and biased beliefs exhibits a Hopf bifurcation at $\beta = 50$. A rational route to randomness (i.e., a bifurcation route to chaos) occurs, with positive largest Lyapunov exponents, when the intensity of choice becomes large.
trend, and phases of “pessimism,” with (small) sudden market crashes, as illustrated in Figure 4.3. In fact, in the ABS, prices are characterized by evolutionary switching between the fundamental value and temporary speculative bubbles. Prices deviate persistently from their fundamental value; therefore it can be said that prices are excessively volatile. In this sense the market is inefficient. But are these deviations easy to predict? Even in the simple, stylized four-type example in the purely deterministic chaotic case, the timing and direction of the temporary bubbles seem hard to predict, but once a bubble has started, the collapse of the bubble seems predictable. In the presence of (small) noise, however, the situation is quite different, as illustrated in Figure 4.3 (top right): The timing, the direction and the collapse of the bubble all seem hard to predict.

To stress this point further, we investigate this (un)predictability by employing a so-called nearest neighbor forecasting method to predict the returns, at lags 1 to 20 for the purely chaotic as well as for several noisy chaotic time series, as illustrated in Figure 4.5.12 Nearest-neighbor forecasting looks for patterns in the past that are close to the most recent pattern and then predicts the average value following all nearby past patterns. According to Takens’ embedding theorem, this method yields good forecasts for deterministic chaotic systems.13 Figure 4.5 shows that as the noise level increases, the forecasting performance of the nearest-neighbor method quickly deteriorates. Therefore, in our simple nonlinear evolutionary ABS with noise, it is hard to make good forecasts of future returns and to predict when prices will return to fundamental value. Our simple nonlinear ABS with small noise thus captures some of the intrinsic unpredictability of asset returns also present in real markets, and in terms of predictability the market is close to being efficient.

4.3.5. Wealth Accumulation

The evolutionary dynamics in an ABS are driven by realized short-run profits, and chartists strategies survive in a world driven by short-run profit opportunities. In this subsection, we briefly look at the accumulated wealth in an ABS. Recall that accumulated wealth for strategy type \( h \) is given by

\[
W_{h,t+1} = RW_{ht} + (p_{t+1} + y_{t+1} - Rp_t)z_{ht}
\]  

(4.29)

The first term represents wealth growth due to the risk-free asset, while the last term represents wealth growth (or decay) due to investments in the risky asset. Because of market clearing, the average net inflow of wealth due to investment in the risky asset is given by

\[
\sum_h n_{ht}z_{ht}(p_{t+1} + y_{t+1} - Rp_t) = z^s(p_{t+1} + y_{t+1} - Rp_t)
\]  

(4.30)

12We would like to thank Sebastiano Manzan for providing this figure.
13See Kantz and Schreiber (1997) for an extensive treatment of nonlinear time-series analysis and forecasting techniques.
FIGURE 4.5  Forecasting errors for nearest-neighbor method applied to chaotic and noisy chaotic returns series for different noise levels in the four-type adaptive belief system. All returns series have close to 0 autocorrelations at all lags. The benchmark case of prediction by the mean 0 is represented by the horizontal line at the normalized prediction error 1. Nearest-neighbor forecasting applied to the purely deterministic chaotic series leads to much smaller forecasting errors at all prediction horizons, 1–20 (bottom graph). A noise level of, say, 10% means that the ratio of the variance of the noise $\epsilon_t$ and the variance of the deterministic price series is $1/10$. As the noise level increases, the graphs shift upward, indicating that prediction errors increase. Small dynamic noise thus quickly deteriorates forecasting performance.

This is the average risk premium required by the population of investors to hold the risky asset. In the special case $\zeta^s = 0$ the risk premium is 0 and on average wealth of each strategy grows at the risk-free rate.

Figure 4.6 shows the development of prices and wealth of each strategy in the three-type and four-type examples of Subsections 4.3.2 and 4.3.3. Prices fluctuate around the fundamental price. For the three-type example, the wealth accumulated by each of the three strategies—fundamentalists, optimistic biased, and pessimistic biased—grows over time, at an equal rate. Recall that in the three-type example, for an infinite intensity of choice $\beta$, the system converges to a stable four-cycle with average profits equal for all three strategies. At each time $t$, profits of fundamentalists are always between profits of optimists and pessimists, but on average all profits are (almost) equal, and thus accumulated wealth grows at the same rate.\textsuperscript{14}

\textsuperscript{14}For finite intensity of choice, for example, $\beta = 3000$ as in Figure 4.6, wealth of the three-types grows at almost the same rate. For initial states chosen, as in Figure 4.6, wealth of the optimistic types slightly dominates the other two types.
In the four-type example, trend-following strategies are profitable during temporary bubbles. Fundamentalists suffer losses during temporary bubbles, but these losses are limited. When the bubble bursts, fundamentalists make large profits while trend followers suffer from huge losses. On average, accumulated wealth of fundamentalists increases while wealth of chartists decreases, as illustrated in Figure 4.6.
The wealth in Eq. 4.29 corresponds to the accumulated wealth of a trader who always uses strategy $h$. How would a switching strategy perform in a heterogeneous market? Figure 4.6 (bottom panel) illustrates an example of an ABS with five strategies, where a switching strategy has been added to the four-type ABS. The fifth switching strategy is endogenous in the five-type ABS and thus affects the realized market price in the same way as the other four strategies. The switching strategy always picks the best of the other four strategies, according to last period’s realized profits, conditional on how far the price deviates from the fundamental benchmark. In the simulation, when the price deviation becomes larger than a threshold parameter ($\delta = 0.5$), the switching strategy switches back to the fundamental strategy to avoid losses when the bubble collapses.

Figure 4.6 (bottom row) illustrates two features of the five-type ABS. First, due to the presence of the switching strategy, the amplitude of price fluctuations (bottom row, left plot) is somewhat smaller than in the four-type ABS. This is caused by the switching strategy switching back to the fundamental strategy when the price deviation exceeds the threshold. Second, the accumulated wealth of the switching strategy outperforms all other strategies, including the fundamental strategy—see Figure 4.6, bottom row, right plot. Notice that the two best strategies, the switching strategy and the fundamental strategy, also require the most information. The trend-following strategies only use publically available information on past prices. The fundamental strategy uses fundamental information as well as information about competing strategies in the market and their performance.

In the ABS evolutionary framework, agents switch strategies based on short-run realized profits. In the long run, a fundamental strategy often accumulates more wealth than trend-following rules. However, fundamental strategies suffer from losses during temporary bubbles when prices persistently deviate from fundamentals and may therefore suffer from “limits of arbitrage” (Shleifer and Vishny, 1997). Fundamentalists can stabilize price fluctuations but only if they are not limited by borrowing constraints or limits of arbitrage. In the long run, a simple switching strategy may accumulate more wealth than a fundamental or technical trading strategy. The fact that a simple switching strategy performs better in a heterogeneous market shows that the ABS model is behaviorally consistent. Agents have an incentive to keep switching strategies.

The switching strategy is very risky, however, because it requires good knowledge of the underlying fundamental and good market timing to “get off the bubble before it bursts.” Interestingly, Zwart et al. (2007) provided empirical evidence, analyzing 15 emerging market currencies over the period from 1995 to 2006, that a combined strategy with time-varying weights may generate economically and statistically significant returns, after accounting for transaction costs. Their strategy is based on a combination of fundamental information on the deviation from purchasing power parity and the real interest rate differential and chartist information from moving average trading rules, with time-varying weights determined by relative performance over the past year.

Recall that these strategies can be formulated without knowledge of the fundamental price; see Footnote 4.
4.3.6. Extensions


4.4. MANY TRADER TYPES

In most heterogeneous agent models (HAMs) in the literature, the number of trader types is small—usually only two, three, or four types are considered that use simple fundamentalist or chartist strategies. Generally, analytical tractability can only be obtained at the cost of restricting a HAM to just a few types. However, Brock, Hommes, and Wagener (2005) have developed a theoretical framework to study evolutionary markets with many different trader types. In this subsection, we discuss their notion of large type limit (LTL), a simple, low-dimensional approximation of an evolutionary ABS with many trader types. The LTL can be developed in a fairly general market-clearing setting, but here we focus on its application to the asset-pricing model with heterogeneous beliefs.

Recall from Eq. 4.1 that in the asset market with \( H \) different trader types, the equilibrium price (in deviations \( x_t \) from the fundamental benchmark) is given by

\[
x_t = \frac{1}{1 + r} \sum_{h=1}^{H} n_{ht} f_{ht}
\]

16 In the artificial market of Levy et al. (1994), asset demand is also derived from CRRA utility.
17 Another related stochastic model with heterogeneous agents and endogenous strategy switching similar to the ABS has been introduced in Föllmer et al. (2005). Scheinkman and Xiong (2004) review related stochastic financial models with heterogeneous beliefs and short-sale constraints. Macro models with heterogeneous expectations have been studied, for instance, in Branch and Evans (2006) and Branch and McGough (2008).
Using the multinomial logit probabilities (Eq. 4.13) for the fractions \( n_{ht} \), we get

\[
x_t = \frac{1}{1 + r} \frac{\sum_{h=1}^{H} e^{\beta U_{h,t-1} f_{ht}}}{\sum_{h=1}^{H} e^{\beta U_{h,t-1}}} \tag{4.32}
\]

It is assumed that prediction and fitness functions take the form

\[
f_{ht} = f(x, \lambda, \vartheta_h) \quad \text{and} \quad U_{ht} = U(x, \lambda, \vartheta_h)
\]

respectively, where \( x = (x_{t-1}, x_{t-2}, \ldots) \) is a vector of lagged deviations from the fundamental, \( \lambda \) is a structural parameter vector (e.g., containing the risk-free interest rate \( r \), the risk-aversion parameter \( a \), the intensity of choice \( \beta \), etc.), and \( \vartheta_h \) is a multidimensional variable that characterizes the belief type \( h \).

The equilibrium Eq. 4.32 determines the evolution of the system with \( H \) trader types; this information is coded in the evolution map \( \varphi_H(x, \lambda, \vartheta) \):

\[
\varphi_H(x, \lambda, \vartheta) = \frac{1}{1 + r} \frac{\sum_{h=1}^{H} e^{\beta U(x, \lambda, \vartheta_h) f(x, \lambda, \vartheta_h)}}{\sum_{h=1}^{H} e^{\beta U(x, \lambda, \vartheta_h)}} \tag{4.33}
\]

where \( \vartheta = (\vartheta_1, \ldots, \vartheta_H) \). At the beginning of the market, a large number \( H \) of beliefs \( \vartheta_h \) is sampled from a general distribution of initial beliefs. For example, all forecasting rules may be drawn from a linear class of rules with \( L \) lags,

\[
f_t(\vartheta_0) = \delta_{00} + \delta_{01} x_{t-1} + \delta_{02} x_{t-2} + \cdots + \delta_{0L} x_{t-L} \tag{4.34}
\]

with \( \delta_{0h}, h = 0, \ldots, L \), drawn from a multivariate normal distribution.

The evolution map \( \varphi_H \) in Eq. 4.33 determines the dynamical system corresponding to an asset market with \( H \) different belief types. When the number of trader types \( H \) is large, this dynamical system contains a large number of stochastic variables \( \delta = (\delta_1, \ldots, \delta_H) \), where the \( \delta_h \) are IID, with distribution function \( F_\mu \). At the beginning of the market, \( H \) belief types are drawn from this distribution, and they then compete against each other. The distribution function of the stochastic belief variable \( \delta_h \) depends on a multidimensional parameter \( \mu \), called the belief parameter. This setup allows us to vary the population out of which the individual beliefs are sampled at the beginning of the market.

Observe that both the denominator and the numerator of the evolution map \( \varphi_H \) in Eq. 4.33 may be divided by the number of trader types \( H \) and thus may be seen as sample means. The evolution map \( \psi \) of the LTL is then obtained by replacing sample means in the evolution map \( \varphi_H \) by population means:

\[
\psi(x, \lambda, \mu) = \frac{1}{1 + r} \frac{E_\mu \left[ e^{\beta U(x, \lambda, \vartheta_0)} f(x, \lambda, \vartheta_0) \right]}{E_\mu \left[ e^{\beta U(x, \lambda, \vartheta_0)} \right]} \tag{4.35}
\]

\[
= \frac{1}{1 + r} \int e^{\beta U(x, \lambda, \vartheta_0)} f(x, \lambda, \vartheta_0) d\nu_\mu
\]

where \( \vartheta_0 \) is a stochastic variable, distributed in the same way as the \( \delta_h \), with density \( \nu_\mu \). The structural parameter vector \( \lambda \) of the evolution map \( \varphi_H \) and of the LTL evolution
map $\psi$ coincide. However, whereas the evolution map $\varphi_H$ in Eq. 4.33 of the heterogeneous agent system contains $H$ randomly drawn multidimensional stochastic variables $\vartheta_h$, the LTL evolution map $\psi$ in Eq. 4.35 only contains the belief parameter vector $\mu$ describing the joint probability distribution. Taking an LTL thus leads to a huge reduction in stochastic belief variables.

According to the LTL theorem of Brock et al. (2005), as the number $H$ of trader types tends to infinity, the $H$-type evolution map $\varphi_H$ converges almost surely to the LTL map $\psi$. This implies that the corresponding LTL dynamical system is a good approximation of the dynamical behavior in a heterogeneous asset market when the number of belief types $H$ is large. In particular, all generic and persistent dynamic properties will be preserved with high probability. For example, if the LTL map exhibits a bifurcation route to chaos for one of the structural parameters, then, if the number of trader types $H$ is large, the $H$-type system also exhibits such a bifurcation route to chaos with high probability.

A straightforward computation using moment-generating functions shows that, for example, in the case of linear forecasting rules (Eq. 4.34) with three lags ($L = 3$), the corresponding LTL becomes a 5-D nonlinear system given by

$$
(1 + r)x_t = \mu_0 + \mu_1 x_{t-1} + \mu_2 x_{t-2} + \mu_3 x_{t-3} + \eta(x_{t-1} - Rx_{t-2} + a\sigma^2 z^s) \tag{4.36}
$$

$$
(\sigma_0^2 + \sigma_1^2 x_{t-1} x_{t-3} + \sigma_2^2 x_{t-2} x_{t-4} + \sigma_3^2 x_{t-3} x_{t-5})
$$

where $\eta = \beta/(a\sigma^2)$. The simplest special case of Eq. 4.36 that still leads to interesting dynamics is obtained when all $\vartheta_{0k} = 0, 1 \leq k \leq d$, that is, when the forecasting function (Eq. 4.34) is purely biased: $f_i(\vartheta_0) = \vartheta_{00}$. The LTL then simplifies to the linear system

$$
Rx_t = \mu_0 + \eta \sigma_0^2 (x_{t-1} - Rx_{t-2} + a\sigma^2 z^s) \tag{4.37}
$$

This simplest case already provides insight into the (in)stability of the (fundamental) steady state in an evolutionary system with many trader types. When there is no intrinsic mean bias (i.e., when the mean of the biases $\vartheta_{00}$ equals 0 (i.e., $\mu_0 = 0$) and the risk premium is zero ($z^s = 0$)), the steady state of the LTL (Eq. 4.37) coincides with the fundamental: $x^* = 0$. When the mean bias and risk premium are both positive (negative), the steady-state deviation $x^*$ will be positive (negative) so that the steady state will be above (below) the fundamental. The natural bifurcation parameter tuning the (in)stability of the system is $\eta \sigma_0^2 = \beta \sigma_0^2 / a\sigma^2$. We see that instability occurs if and only if $\eta$ is increased beyond the bifurcation point $\eta_c = 1/\sigma_0^2$. Therefore this simple case already suggests mechanisms that may destabilize the evolutionary system: an increase in choice intensity $\beta$ for evolutionary selection, a decrease in risk aversion $a$, a decrease in conditional variance of excess returns $\sigma^2$, or an increase in the diversity of purely biased beliefs $\sigma_0^2$. All these forces can push $\eta$ beyond $\eta_c$, thereby triggering instability of the (fundamental) steady state.

For the LTL in Eq. 4.36, in the case of linear forecasting rules with three lags, a bifurcation route to chaos, with asset prices fluctuating around the unstable fundamental steady state, occurs when $\eta$ is increased. This shows that a rational route to randomness can occur in an asset market with many different trader types, when traders...
FIGURE 4.7 Bifurcation diagram in the \((\eta, \mu_1)\) parameter plane for LTL (Eq. 4.36), where \(\mu_1\) represents the mean of the first-order stochastic trend variable \(\delta_{01}\) in the forecasting rule (Eq. 4.34). For \(\mu_0 = a\sigma^2 = 0\), with \(\mu_0\) the mean of the constant \(\delta_{00}\) in the forecasting rule (Eq. 4.34), the LTL is symmetric and thus nongeneric (dotted curves); when \(\mu_0 \neq 0\), the LTL is nonsymmetric and generic. The diagram shows Hopf (H), period-doubling (PD), pitchfork (PF), and saddle-node (SN) bifurcation curves in the \((\eta, \mu_1)\) parameter plane, with other parameters fixed at \(R = 1.01, z^t = 0, \mu_2 = \mu_3 = 0, \sigma_0 = \sigma_1 = \sigma_2 = 1, \) and \(\sigma_3 = 0\). Between the H and PD curves (and the PF curve when \(\mu_0 = 0\)), there is a unique, stable steady state. This steady state becomes unstable when crossing the H or the PD curve. Above the PF curve or the SN curve, the system has three steady states. The PF curve is nongeneric and only arises in the symmetric case with mean bias \(\mu_0 = 0\). When the symmetry is broken by perturbing the mean bias to \(\mu_0 = -0.1\), the PF curve “breaks” into generic Hopf and SN curves.

become increasingly sensitive to differences in fitness (i.e., an increase in the intensity of choice \(\beta\)) or traders become less risk averse (i.e., a decrease of the coefficient of risk-aversion \(a\)). In general, in a many-trader-type evolutionary world, fundamentalists will not drive out all other types and asset prices need not converge to their fundamental values.

Figure 4.7 shows a two-dimensional bifurcation diagram in the \((\eta, \mu_1)\) parameter plane, where \(\mu_1\) represents the mean of the first-order stochastic trend variable \(\delta_{01}\) in the forecasting rule (Eq. 4.34). Recall that \(\mu_0\) is the mean of the constant term \(\delta_{00}\) in the forecasting rule (Eq. 4.34); it models the “mean bias” of the trader type. When \(\mu_0 = 0\) and \(a\sigma^2z_s = 0\) (expressing that the risk premium is zero), the LTL is symmetric with respect to the fundamental steady state.

In the symmetric case (dotted lines in Figure 4.7), for parameters taking values in the region enclosed by the Hopf, period-doubling (PD), and pitchfork (PF) bifurcation curves, the fundamental steady state is unique and stable.\(^{18}\) As the parameters cross the PF curve, two additional nonfundamental steady states are created, one above and

\(^{18}\)See Appendix 4.1 for a brief introduction to bifurcation theory, with Appendix 4.2 providing simple examples of the saddle-node, period doubling, pitchfork, and Hopf bifurcations.
one below the fundamental. Another route to instability occurs when crossing the Hopf curve, where the fundamental steady state becomes unstable and a (stable) invariant circle with periodic or quasi-periodic dynamics is created. The pitchfork bifurcation curve is \textit{nongeneric} and occurs only in the symmetric case. When the symmetry is broken by a nonzero mean bias $\mu_0 \neq 0$, as illustrated in Figure 4.7 (bold curves) for $\mu_0 = -0.1$, the PF curve disappears and “breaks” into two generic codimension bifurcation curves: a Hopf and a saddle-node (SN) bifurcation curve. When crossing the SN curve from below, two additional steady states are created, one stable and one unstable. Notice that, as illustrated in Figure 4.7, when the perturbation is small (in the figure, $\mu_0 = -0.1$), the SN and the Hopf curves are close to the PF and the Hopf curves (dotted lines) in the symmetric case. In this sense the bifurcation diagram depends continuously on the parameters, and it is useful to consider the symmetric LTL as an “organizing” center to study bifurcation phenomena in the generic, nonsymmetric LTL.

The most relevant case from an economic viewpoint arises when the mean $\mu_1$ of the first-order coefficient $\delta_{01}$ in the forecasting rule (Eq. 4.34) satisfies $0 \leq \mu_1 \leq 1$. In that case, the (fundamental) steady state loses stability in a Hopf bifurcation as $\eta$ increases. Figure 4.8 illustrates the dynamical behavior of the LTL as the parameter $\eta$ further increases.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures/figure4_8.png}
\caption{Attractors in the phase space for the 5-D LTL with parameters $R = 1.01$, $\varepsilon^* = 0$, $\mu_0 = 0$, $\mu_1 = 0$, $\mu_2 = \mu_3 = 0$, and $\sigma_0 = \sigma_1 = \sigma_2 = \sigma_3 = 1$: (a) immediately after the Hopf bifurcation (quasi-)periodic dynamics on a stable invariant circle occurs; (b–c) after a Hopf bifurcation (quasi-)periodic dynamics on a stable invariant torus occurs; and (d–f) breaking up of the invariant torus into a strange attractor.}
\end{figure}
increases. After the Hopf bifurcation periodic and quasi-periodic dynamics on a stable invariant circle occur, and for increasing values of $\eta$, a bifurcation route to strange attractors occurs. Figure 4.8 thus presents numerical evidence of the occurrence of a rational route to randomness, that is, a bifurcation route to strange attractors as the intensity of choice to switch forecasting strategies increases. If such rational routes to randomness occur for the LTL, the LTL convergence theorem implies that in evolutionary systems with many trader types, rational routes to randomness occur with high probability.

Diks and van der Weide (2003, 2005) have generalized the notion of LTL and introduced so-called continuous belief systems (CBS), where the beliefs of traders are distributed according to a continuous density function. The beliefs distribution function and the equilibrium prices coevolve over time. The LTL theory discussed here as well as its extensions can be used to form a bridge between an analytical approach and the literature on evolutionary artificial market simulation models reviewed in LeBaron (2000, 2006). See also Anufriev et al. (2008) for a recent application of a LTL in a macroeconomic model with heterogeneous expectations.

4.5. EMPIRICAL VALIDATION

In this section we discuss the empirical validity of the asset pricing model with heterogeneous beliefs. There is already a large literature on heterogeneous agent models replicating many of the important stylized facts of financial time series on short time scales (say, daily or higher frequency), such as fat tails and long memory in the returns distribution and clustered volatility. Examples of HAMs able to replicate stylized facts of financial markets include, for example, Brock and LeBaron (1996), Arthur et al. (1997), Brock and Hommes (1997b), Youssefmir and Huberman (1997), LeBaron et al. (1999), Lux and Marchesi (1999, 2000), Farmer and Joshi (2002), Kirman and Teyssière (2002), Hommes (2002), Iori (2002), Cont and Bouchaud (2000), and Gaunersdorfer and Hommes (2007). The recent survey by Lux (2009) contains an extensive look at behavioral interacting agent models mimicking the stylized facts of asset returns. We have already seen examples of simple heterogeneous agent models mimicking temporary bubbles and crashes. In this section we discuss how these qualitative features match observed bubbles and crashes in real markets.

Empirical validation and estimation of HAMs on economic or financial data are still in their infancy. An early attempt has already been made by Shiller (1984) who presented a HAM with smart money traders having rational expectations versus ordinary investors (whose behavior is in fact not modeled at all). Shiller estimated the fraction of smart money investors over the period 1900 to 1983 and found considerable fluctuations of the fraction over a range between 0% and 50%. Baak (1999) and Chavas (2000) estimated HAMs on hog and beef market data and found evidence for heterogeneity of expectations. Winker and Gilli (2001) and Gilli and Winker (2003) estimated the model of Kirman (1991, 1993) with fundamentalists and chartists, using the daily DM-US$ exchange rates from 1991 to 2000. Their estimated parameter values correspond to a bimodal distribution of agents. Westerhoff and Reitz (2003) also
estimated an HAM with fundamentalists and chartists to exchange rates and found considerable fluctuations of the market impact of fundamentalists. Alfarno et al. (2005) estimated an agent-based herding model where agents switch between fundamentalist and chartist strategies. Branch (2004) estimated a model with heterogeneous beliefs and time-varying fractions, using survey data on inflation expectations. In this section, we discuss the estimation of a simple two-type asset-pricing model with heterogeneous beliefs, as discussed in Section 4.2, on yearly S&P 500 data, 1871 to 2003, as done in Boswijk et al. (2007). As we will see, this simple two-type model can, for example, explain the dot-com bubble in the late 1990s and the subsequent crash starting at the end of 2000.19

4.5.1. The Model in Price-to-Cash Flows

In the previous sections, the dividend process of the risky asset has been assumed to be stationary. To estimate the model using yearly data of more than a century, the dividend process has to be taken growing over time and thus nonstationary. To estimate a simple two-type model, Boswijk et al. (2007) therefore reformulated the model in terms of price-to-cash flows. Recall from Eq. 4.5 that, under the assumption of zero net supply of the risky asset, the equilibrium pricing equation is

\[
p_t = \frac{1}{1 + r} \sum_{h=1}^{H} n_{h,t} E_{h,t} (p_{t+1} + y_{t+1})
\]

or equivalently

\[
r = \sum_{h=1}^{H} n_{h,t} \frac{E_{h,t} [p_{t+1} + y_{t+1} - p_t]}{p_t}
\]

In equilibrium the average required rate of return for investors to hold the risky asset equals the discount rate \( r \). In the estimation of the model, the discount rate \( r \) has been set equal to the sum of the (risk-free) interest rate and the required risk premium on stocks. A simple, nonstationary process that fits cash flow data (dividends or earnings) well is a stochastic process with a constant growth rate. More precisely, assume that \( \log y_t \) is a Gaussian random walk with drift; that is,

\[
\log y_{t+1} = \mu + \log y_t + \nu_{t+1} \quad \nu_{t+1} \sim \text{i.i.d. } N(0, \sigma_\nu^2)
\]

which implies

\[
\frac{y_{t+1}}{y_t} = e^{\mu + \nu_{t+1}} = e^{\mu + \frac{1}{2} \sigma_\nu^2} e^{\nu_{t+1}} e^{-\frac{1}{2} \sigma_\nu^2} = (1 + g) \varepsilon_{t+1}
\]

19Van Norden and Schaller (1999) estimate a nonlinear time-series switching model with two regimes, an explosive and a collapsing bubble regime, with the probability of being in the explosive regime depending negatively on the relative absolute deviation of the bubble from the fundamental. Brooks and Katsaris (2005) extend this model to three regimes, adding a third dormant bubble regime where the bubble grows at the required rate of return without explosive expectations.
where \( g = \mu + \frac{1}{2} \sigma^2 - 1 \) and \( \epsilon_{t+1} = \epsilon_{t+1} + \frac{1}{2} \sigma^2 \), so that \( E_t(\epsilon_{t+1}) = 1 \). As before, we assume that all types have correct beliefs on the cash flow; that is,

\[
E_{h,t}[y_{t+1}] = E_t[y_{t+1}] = (1 + g) y_t, \quad E_t[\epsilon_{t+1}] = (1 + g) y_t
\]

(4.42)

Since the cash flow is an exogenously given stochastic process, it seems natural to assume that agents have learned the correct beliefs on next period’s cash flow \( y_{t+1} \). In particular, boundedly rational agents can learn about the constant growth rate by, for example, running a simple regression of \( \log(y_t/y_{t-1}) \) on a constant.

In contrast, prices are determined endogenously and are affected by expectations about next period’s price. In a heterogeneous world, agreement about future prices therefore seems more unlikely than agreement about future cash flows. Therefore we assume homogeneous beliefs about future cash flow but heterogeneous beliefs about future prices. The pricing Eq. 4.38 can be reformulated in terms of a price-to-cash-flow (P/Y) ratio, \( \delta_t = p_t/y_t \), as

\[
\delta_t = \frac{1}{R^*} \left( 1 + \sum_{h=1}^{H} n_{h,t} E_{h,t}[\delta_{t+1}] \right) \quad R^* = \frac{1 + r}{1 + g}
\]

(4.43)

In the special case, when all agents have rational expectations, the equilibrium pricing Eq. 4.38 simplifies to \( p_t = (1/(1 + r))E_t(p_{t+1} + y_{t+1}) \). It is well known that, in the case of a constant discount rate \( r \) and a constant growth rate \( g \) for dividends, according to the static Gordon growth model (Gordon, 1962), the rational expectations fundamental price, \( p^*_t \), of the risky asset is given by

\[
p^*_t = \frac{1 + g}{r - g} y_t \quad r > g
\]

(4.44)

Equivalently, in terms of price-to-cash-flow ratios, the fundamental is

\[
\delta^*_t = \frac{p^*_t}{y_t} = \frac{1 + g}{r - g} \equiv m
\]

(4.45)

We refer to \( p^*_t \) as the fundamental price and to \( \delta^*_t \) as the fundamental P/Y ratio. When all agents are rational, the pricing Eq. 4.43 in terms of the P/Y ratio, \( \delta_t = p_t/y_t \), becomes

\[
\delta_t = \frac{1}{R^*} \left( 1 + E_t[\delta_{t+1}] \right)
\]

(4.46)

\(^{20}\)Barberis et al. (1998) consider a model in which agents are affected by psychological biases in forming expectations about future cash flows. In particular, agents may overreact to good news about economic fundamentals because they believe that cash flows have moved into another regime with higher growth. Their model is able to explain continuation and reversal of stock returns.

\(^{21}\)In what follows we use either price-to-dividend (P/D) or price-to-earnings (P/E) ratios and use the general notation P/Y for price-to-cash flows.
In terms of the deviation from the fundamental ratio, \( x_t = \delta_t - \delta^*_t = \delta_t - m \), this simplifies to

\[
x_t = \frac{1}{R^*} E_t[x_{t+1}]
\]

(4.47)

Under heterogeneity in expectations, the pricing Eq. 4.43 is expressed in terms of \( x_t \) as

\[
x_t = \frac{1}{R^*} \sum_{h=1}^{H} n_{h,t} E_{h,t}[x_{t+1}]
\]

(4.48)

**Heterogeneous Beliefs**

The expectation of belief type \( h \) about next period P/Y ratio is expressed as

\[
E_{h,t}[\delta_{t+1}] = E_t[\delta^*_t] + f_h(x_{t-1}, \ldots, x_{t-L}) = m + f_h(x_{t-1}, \ldots, x_{t-L})
\]

(4.49)

where \( \delta^*_t \) represents the fundamental price-to-cash-flow ratio P/Y, \( E_t[\delta^*_t] = m \) is the rational expectation of the P/Y ratio available to all agents, \( x_t \) is the deviation of the P/Y ratio from its fundamental value, and \( f_h(\cdot) \) represents the expected transitory deviation of the P/Y ratio from the fundamental value, depending on \( L \) past deviations. The information available to investors at time \( t \) includes present and past cash flows and past prices. In terms of deviations from the fundamental P/Y ratio, \( x_t \), we get

\[
E_{h,t}[x_{t+1}] = f_h(x_{t-1}, \ldots, x_{t-L})
\]

(4.50)

Note again that the rational expectations fundamental benchmark is nested in the heterogeneous agent model as a special case when \( f_h \equiv 0 \) for all types \( h \). We can express Eq. 4.48 as

\[
R^* x_t = \sum_{h=1}^{H} n_{h,t} f_h(x_{t-1}, \ldots, x_{t-L})
\]

(4.51)

From this equilibrium equation it is clear that the adjustment toward the fundamental P/Y ratio will be slow if a majority of investors has persistent beliefs about it.

**Evolutionary Selection of Expectations**

In addition to the empirical evidence of persistent deviations from fundamentals there is also significant evidence of time variation in the sentiment of investors. This has been documented, for example, by Shiller (1987, 2000) using survey data. In the model considered here, agents are boundedly rational and switch between different forecasting strategies according to recently realized profits. We denote by \( \pi_{h,t-1} \) the realized profits
of type $h$ at the end of period $t - 1$, given by (see Eq. 4.14):

$$\pi_{h,t-1} = R_{t-1} z_{h,t-2} = R_{t-1} \frac{E_{h,t-2}[R_{t-1}]}{aV_{t-2}[R_{t-1}]} \quad (4.52)$$

where $R_{t-1} = p_{t-1} + y_{t-1} - (1 + r)p_{t-2}$ is the realized excess return at time $t - 1$ and $z_{h,t-2}$ is the demand of the risky asset by belief type $h$, as given in Eq. 4.3, formed in period $t - 2$.

As before, we assume that the beliefs about the conditional variance of excess returns are the same for all types and equal to fundamentalists beliefs about conditional variance, that is,

$$V_{h,t-2}[R_{t-1}] = V_{t-2}^*[P_{t-1}^* + y_{t-1} - (1 + r)P_{t-2}^*] = y_{t-2}^2 \eta^2 \quad (4.53)$$

where $\eta^2 = (1 + m)^2(1 + g)^2 V_{t-2}[\varepsilon_{t-1}]$, with $\varepsilon_t$ IID noise driving the cash flow. The fitness measure can be rewritten in terms of the deviation $x_t = \delta_t - m$ of the P/Y ratio from its fundamental value, with $m = (1 + g)/(r - g)$, as

$$\pi_{h,t-1} = \frac{(1 + g)^2}{a\eta^2} (x_{t-1} - R^* x_{t-2}) \left( E_{h,t-2}[x_{t-1}] - R^* x_{t-2} \right) \quad (4.54)$$

This fitness measure has a simple, intuitive explanation in terms of forecasting performance for next period’s deviation from the fundamental. A positive demand $z_{h,t-2}$ may be seen as a bet that $x_{t-1}$ would go up more than what was expected on average from $R^* x_{t-2}$. (Note that $R^*$ is the growth rate of rational bubble solutions.) The realized fitness $\pi_{h,t-1}$ of strategy $h$ is the realized profit from that bet and it will be positive if both the realized deviation $x_{t-1} > R^* x_{t-2}$ and the forecast of the deviation $E_{h,t-2}[x_{t-1}] > R^* x_{t-2}$. More generally, if both the realized absolute deviation $|x_{t-1}|$ and the absolute predicted deviation $|E_{h,t-2}[x_{t-1}]|$ to the fundamental value are larger than $R^*$ times the absolute deviation $|x_{t-2}|$, strategy $h$ generates positive realized fitness. In contrast, a strategy that wrongly predicts whether the asset price mean reverts back toward the fundamental value or moves away from it generates a negative realized fitness.

At the beginning of period $t$, investors compare the realized relative performances of the various strategies and withdraw capital from those that performed poorly and move it to better strategies. The fractions $n_{h,t}$ evolve according to a discrete choice model with multinomial logit probabilities, that is (see Eq. 4.13),

$$n_{h,t} = \frac{\exp[\beta \pi_{h,t-1}]}{\sum_{k=1}^{H} \exp[\beta \pi_{k,t-1}]} = \frac{1}{1 + \sum_{k \neq h} \exp[-\beta \Delta \pi_{t-1}^{h,k}]} \quad (4.55)$$

where $\beta > 0$ is the intensity of choice as before, and $\Delta \pi_{t-1}^{h,k} = \pi_{h,t-1} - \pi_{k,t-1}$ denotes the difference in realized profits of belief type $h$ compared to type $k$. 
4.5.2. Estimation of a Simple Two-Type Example

Consider the case of two types, both predicting next period’s deviation by extrapolating past realizations in a linear fashion, that is,\(^{22}\)

\[ E_{h,t}[x_{t+1}] = f_h(x_{t-1}) = \varphi_h x_{t-1} \tag{4.56} \]

The dynamic asset-pricing model with two types can then be written as

\[ R^* x_t = n_t \varphi_1 x_{t-1} + (1 - n_t) \varphi_2 x_{t-1} + \epsilon_t \tag{4.57} \]

where \(\varphi_1\) and \(\varphi_2\) denote the coefficients of the two belief types, \(n_t\) represents the fraction of investors that belong to the first type of traders and \(\epsilon_t\) represents a disturbance term. The value of the parameter \(\varphi_h\) can be interpreted as follows: If it is positive and smaller than 1, investors expect the stock price to mean-revert toward the fundamental value. This type of agent represents **fundamentalists** because they expect the asset price to move back toward its fundamental value in the long run. The closer \(\varphi_h\) is to 1, the more persistent are the expected deviations. If the beliefs parameter \(\varphi_h\) is larger than 1, it implies that investors believe the deviation of the stock prices will grow over time at a constant speed. We will refer to this type of agent as a **trend follower**. Note in particular that when one group of investors believes in a strong trend, that is, \(\varphi_h > R^*\), this may cause asset prices to deviate further from their fundamental value. In the case with two types with linear beliefs (Eq. 4.56), the fraction of Type 1 investors is

\[ n_t = \frac{1}{1 + \exp\left\{-\beta^* \left( (\varphi_1 - \varphi_2) x_{t-3} (x_{t-1} - R^* x_{t-2}) \right) \right\}} \tag{4.58} \]

where \(\beta^* = \beta (1 + g)^2 / (\alpha \eta^2)\).

The two-type model (Eqs. 4.57 and 4.58) has been estimated using an updated version of the dataset described in Shiller (1989), consisting of annual observations of the S&P 500 index from 1871 to 2003. Here we present the estimation results with earnings as cash flows, but using dividends as cash flows gives similar results. The valuation ratios are then the P/E ratios.\(^{23}\)

Recall that according to the static Gordon growth model, the fundamental price is given by

\[ p_t^* = m y_t \quad m = \frac{1 + g}{r - g} \tag{4.59} \]

The fundamental value of the asset is a multiple \(m\) of its cash flow, where \(m\) depends on the discount rate \(r\) and the cash-flow growth rate \(g\). The multiple \(m\) can also be

\(^{22}\)In the estimation of the model, higher-order lags turned out to be insignificant, so we focus on the simplest case with only one lag in the function \(f_h(\cdot)\), with \(\varphi_h\) the parameter characterizing the strategy of type \(h\).

\(^{23}\)Since earnings data are noisy, to determine the fundamental valuation we follow the practice of Campbell and Shiller (2005) to smooth earnings by a 10-year moving average.
interpreted as the P/D or the P/E ratios implied by the present value model. Figure 4.9 shows the (log) of yearly S&P 500 data together with the fundamental benchmark as well as their P/E ratios. The figure shows a clear long-term comovement of the stock price and the fundamental value. However, the P/E ratio takes persistent swings away from the constant value predicted by the present-value model. This suggests that the fundamental value does not account completely for the dynamics of stock prices, as was suggested in the early debate on mean reversion by Summers (1986). A survey of the ongoing debate is given in Campbell and Shiller (2005). Here we use the simple

\[ p_t^* = m y_t, \text{ with } m = \frac{1 + g}{1 + r}. \]

(a) shows logs of S&P 500 and the log of the fundamental \( p_t^* \); (b) shows the P/E-ratio of the S&P 500 around the constant fundamental benchmark \( p_t^*/y_t = m \).
constant-growth Gordon model for the fundamental price and estimate the two-type model on deviations from this benchmark.\(^{24}\)

Recall that \( R^* = \frac{1 + r}{1 + g} \), where \( g \) is the constant growth rate of the cash flow and \( r \) is the discount rate equal to the risk-free interest rate plus a risk premium. We use an estimate of the risk premium—the difference between the expected return on the market portfolio of common stocks and the risk-free interest rate—to obtain \( R^* \), as in Fama and French (2002). The risk premium satisfies

\[
RP = g + \frac{y}{p} - i
\]  \( (4.60) \)

where \( g \) is the growth rate of dividends, \( y/p \) denotes the average dividend yield \( y_t/p_{t-1} \) and \( i \) is the risk-free interest rate. For annual data from 1871 to 2003 of the S&P 500, the estimates are \( i = 2.57\% \) and \( RP = 6.56\% \) so that \( r = 9.13\% \) and \( R^* = 1.074.\(^{25}\) The corresponding average P/E ratio is 13.4, as illustrated in Figure 4.9.

Using yearly data of the S&P 500 index from 1871 to 2003, the parameters \((\phi'_1, \phi'_2, \beta^*)\) in the two-type model (Eqs. 4.57 and 4.58) can be estimated by nonlinear least squares. The estimation results are as follows:

\[
R^* x_t = n_t \left\{ 0.80 x_{t-1} \right\} + (1 - n_t) \left\{ 1.097 x_{t-1} \right\} + \hat{\epsilon}_t
\]

\( (0.074) \)  \( (0.052) \)  \( (4.61) \)

\[
n_t = \left\{ 1 + \exp\left[ -7.54(-0.29x_{t-3})(x_{t-1} - R^* x_{t-2}) \right] \right\}^{-1}
\]

\( (4.93) \)

\[
R^2 = 0.77, \quad AIC = 2.23, \quad AIC_{AR(1)} = 2.29, \quad \varphi_{AR(1)} = 0.983,
\]

\[
QLB(4) = 0.94, \quad F^{\text{boot}}(p\text{-value}) = 10.15 (0.011)
\]

The belief coefficients are highly significant and different from each other. On the other hand, the intensity of choice \( \beta^* \) is not significantly different from zero. This is a common result in nonlinear switching-type regression models, where the parameter \( \beta^* \) in the transition function is difficult to estimate and has a large standard deviation because relatively large changes in \( \beta^* \) cause only a small variation of the fraction \( n_t \).

Teräsvirta (1994) argues that this should not be worrying as long as there is significant heterogeneity in the estimated regimes. The nonlinear switching model achieves a lower value for the AIC selection criterion compared to a linear AR(1) model. This suggests that the model is capturing nonlinearity in the data. This is also confirmed by the bootstrap F-test for linearity, which strongly rejects the null hypothesis of linearity

\(^{24}\)The same approach can be used for more general, time-varying fundamental processes. Manzan (2003) shows that a dynamic Gordon model for the fundamental price, where the discount rate \( r \) and/or the growth rate \( g \) are time varying, does not explain the large fluctuations in price-to-cash-flow ratios and in fact yields a fundamental price pattern close to that for the static Gordon model. Boswijk et al. (2007) also estimate a version of the model allowing for time variation in the growth rate of the cash flow and obtain similar results.

\(^{25}\)These estimates are slightly different from Fama and French (2002) because, as in Shiller (1989), we use the CPI index to deflate nominal values.
in favor of the heterogeneous agent model. The residuals of the regression do not show significant evidence of autocorrelation at the 5% significance level.

The estimated coefficient of the first regime is 0.80, corresponding to a half-life of about three years. The first regime can be characterized as fundamentalist beliefs, expecting the asset price to move back toward its fundamental value. In contrast, the second regime has an estimated coefficient close to 1.1, implying that in this regime agents are trend followers, believing the deviation of the stock price to grow over time at a constant speed larger than $R^* \approx 1.074$. At times when the fraction of investors using this belief is equal or close to 1, we have explosive behavior in the P/E ratio. The sentiment of investors switches between a stable fundamentalists regime and a trend-following regime. In normal periods agents consider the deviation a temporary phenomenon and expect it to quickly revert to fundamentals. In other periods, a rapid increase in stock prices not paralleled by improvements in the fundamentals causes losses for fundamentalists and profits for trend followers. Evolutionary pressure will then cause more fundamentalists to become trend followers, thus reinforcing the trend in prices.

Figure 4.10 shows the time series of the fraction of fundamentalists and the average market sentiment, defined as

$$
\varphi_t = \frac{n_t \varphi_1 + (1 - n_t) \varphi_2}{R^*}
$$

(4.62)

It is clear that the fraction of fundamentalists varies considerably over time, with periods in which it is close to 0.5 and other periods in which it is close to either of the extremes 0 or 1. The series of the average market sentiment shows that there is significant time variation between periods of strong mean reversion when the market is dominated by fundamentalist and other periods in which $\varphi_t$ is close to or exceeds 1 and the market is dominated by trend followers. These plots also offer an explanation of the events of the late 1990s: For six consecutive years the trend-following strategy outperformed the fundamentalist strategy and a majority of agents switched to the trend-following strategy, driving the average market sentiment beyond 1 and thus reinforcing the strong price trend. However, at the turn of the market in 2000, the fraction of fundamentalists increased again, approaching 1 and thus contributing to the reversal toward the fundamental value in subsequent years.

The estimation results show that there are two different belief strategies: one in which agents expect continuation of returns and the other in which they expect reversal. We also find that there are some years in which one type of expectation dominates the market. It is clear that the expectation of continuation of positive returns dominated the market in the late 1990s, with the average market sentiment coefficient $\varphi_t$ in Eq. 4.62 larger than 1 in the late 1990s. Despite the awareness of the mispricing, in this period investors were aggressively extrapolating the continuation of the extraordinary performances realized in the previous years. Our approach endogenizes the switching of agents among beliefs. The evolutionary mechanism that relates predictor choice to their past performance is supported by the data. It also confirms previous evidence that pointed in this direction.
Based on answers to a survey, Shiller (2000) constructed indices of “bubble expectations” and “investor confidence.” In both cases, he found that the time variation in the indices is well explained by the lagged change in stock prices. Based on a different survey, Fisher and Statman (2002) found that in the late 1990s individual investors had expectations of continuation of recent stock returns while institutional investors were expecting reversals. This is an interesting approach to identifying heterogeneity of beliefs based on the type of investors rather than the type of beliefs. In the view of our model, the bubble in the 1990s was triggered by good news about economic fundamentals (a new Internet technology) and strongly reinforced by trend extrapolating behavior. The bubble was reversed by bad news about economic fundamentals (excessive growth cannot last forever and is not supported by earnings), and the crash was accelerated by switching of beliefs back to fundamentals.
4.5.3. Empirical Implications

In this subsection we discuss some empirical implications of the estimation of our nonlinear evolutionary switching model with heterogeneous beliefs. First, we investigate the response to a positive shock to fundamentals when the asset is overvalued. Second, we address the question concerning the probability that a bubble may resume by considering the evolution of the valuation ratios conditional on data until the end of 2003. These simulation experiments both show the importance of considering nonlinear effects in the dynamics of stock prices.

Response to a Fundamental Shock

We use the estimated parameters to investigate the response of the market valuation to good news. Assume that at the beginning of period $t$ the cash flow increases due to a permanent increase in its growth rate. This implies that the asset has a higher fundamental valuation ratio, but what is the effect on the market valuation? We address this question both for the nonlinear switching model and a linear benchmark. The linear model may be interpreted as a model with a representative agent believing in an average mean reversion toward the fundamental. Assume that at $t - 1$ the fundamental valuation ratio was 15 and the good news at time $t$ drives it to 17. Assume also that the equilibrium price at $t - 1$ was 16. Figure 4.11 shows the valuation ratio dynamics in response to the good news for both the linear and the nonlinear switching models.

The figure shows the average price path over 2000 simulations of the estimated models. There is a clear difference between the linear and the nonlinear models. In the linear case, the positive shock to the fundamental value leads to an immediate increase of the price followed by mean-reversion thereafter. In contrast, for the nonlinear heterogeneous agent model, the pattern that emerges is consistent with the evidence of short-run continuation of positive returns and long-term reversal. After good news, the agents incorporate the news into their expectations and they expect that part of the previous period overvaluation will persist. One group—the trend followers—overreacts and expects a further increase of the price, while the other group—the fundamentalists—expects the price to diminish over time. The equilibrium price at time $t$ overshoots and almost reaches 18. In the following two periods trend followers continue to buy the stock and drive its price and its valuation ratio even higher. Finally, the reversal starts and drives the ratio back to its long-run fundamental value. Initially, the aggressive investors interpret the positive news as a confirmation that the stock overvaluation was justified by forthcoming news. However, the lack of further good news convinces most investors to switch back to the mean-reverting expectations and the stock price is driven back toward its fundamental value.

26 The linear benchmark is, for example, obtained for $\beta = 0$ or equivalently when both fractions $n_t = 1 - n_t = 0.5$ in Eq. 4.57. Hence, in the linear model there is no strategy switching between different types but rather a representative agent with a linear forecasting rule.
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FIGURE 4.11 Average response (over 2000 simulations) to a shock to the fundamental for the linear representative agent model (dashed line) and the nonlinear two-type switching model (dotted line with circles). At period 0 there is a permanent shock to the fundamental price from 15 to 17. The simulation uses the estimated parameter values for the P/D ratio, with a representative agent average belief parameter $\varphi = 0.968$ and heterogeneous agent parameters $\varphi_1 = 0.762$ and $\varphi_2 = 1.135$ for the two types. The nonlinear heterogeneous agent model exhibits short-run continuation of positive returns and long-term reversal.

Will the Bubble Resume?

As a model forecasting exercise, we simulate the evolution of the valuation ratios using the estimated heterogeneous agent model. We then obtain the predicted evolution of the valuation ratio conditional on the value realized at the end of 2003. Innovations are obtained by reshuffling the estimated residuals. Instead of focusing our attention only on the mean or the median of the distribution, we consider the quantiles corresponding to 10%, 30%, 50%, 70%, and 90% probability over 2000 replications of the estimated model in Eq. 4.61 for the P/E ratio. In addition to the quantiles predicted by our nonlinear model, we also plot those predicted by the linear representative agent model. Figure 4.12 shows the 1- to 5-year periods ahead quantiles of the estimated model’s predictive distribution.

The linear model—Figure 4.12(a)—predicts that the valuation ratio reverts toward the mean at all quantiles considered. In contrast, the nonlinear switching model predicts that there is a significant probability that the ratio may increase again as a result of the activation of the trend-following regime. The 70% and 90% quantiles clearly show that the PD ratio may increase again to levels close to 35. Stated differently, our heterogeneous agent model predicts that with probability over 30% the PD ratio may increase to more than 30. Note, however, that the median predicts that the ratio should decrease as implied by the linear mean-reverting model.

Another implication of our model is that if the first (mean-reverting) regime dominates the beliefs of investors, it will enforce a much faster adjustment than predicted by the linear model. This is clear from the bottom quantiles of the distributions. These
simulations show that predictions from a linear, representative agent model versus a nonlinear, heterogeneous agent model are quite different. In particular, extreme events with large deviations from the benchmark fundamental valuation are much more likely in a nonlinear world.

### 4.6. LABORATORY EXPERIMENTS

Asset-pricing models with heterogeneous beliefs exhibit interesting dynamics characterized by temporary bubbles and crashes, triggered by news about fundamentals and reinforced by self-fulfilling expectations and trend-following investment strategies. The previous section focused on the empirical relevance of such models; this section confronts the model with data from laboratory experiments with human subjects. Laboratory experiments are well suited to discipline the class of behavioral modes...
(or heuristics) boundedly rational subjects may use in economic decision making. Here, we discuss a number of “learning to forecast experiments” in which subjects must forecast the price of an asset for which the realized market price is an aggregation of individual expectations.

In real markets, it is hard to obtain detailed information about investors’ individual expectations. One approach is to collect survey data on individual expectations, as done for example by Turnovsky (1970) on expectations about the Consumer Price Index and the unemployment rate during the post-Korean War period. Frankel and Froot (1987a,b, 1990a,b), Allen and Taylor (1990), Ito (1990), and Taylor and Allen (1992) use a survey on exchange rate expectations and conclude that financial practitioners use different forecasting and trading strategies. A consistent finding from survey data is that at short horizons, investors tend to use extrapolative chartists’ trading rules, whereas at longer horizons investors tend to use mean-reverting fundamentalists’ trading rules. Shiller (1987, 1990, 2000) analyzes surveys on expectations about stock market prices and real estate prices and finds evidence for time variation in investors’ sentiment; see also Vissing-Jorgensen (2003).

Laboratory experiments with human subjects provide an alternative, complementary approach to study the interaction of individual expectations and the resulting aggregate outcomes. An important advantage of the experimental approach is that the experimenter has full control over the underlying economic fundamentals. Surprisingly little experimental work has focused on expectation formation in markets. Williams (1987) considers expectation formation in an experimental double auction market that varies from period to period by small shifts in the market-clearing price. Participants predict the mean contract price for four or five consecutive periods. The participant with the lowest forecast error earns $1.00.

In Smith, Suchanek, and Williams (1998), expectations and the occurrence of speculative bubbles are studied in an experimental asset market. In a series of papers, Marimon, Spear, and Sunder (1993) and Marimon and Sunder (1993, 1994, 1995) studied expectation formation in inflationary overlapping generations’ economies. Marimon, Spear, and Sunder (1993) find experimental evidence for expectationally driven cycles and coordination of beliefs on a sunspot two-cycle equilibrium but only after agents have been exposed to exogenous shocks of a similar kind. Marimon and Sunder (1995) present experimental evidence that a “simple” rule, such as a constant growth of the money supply, can help coordinate agents’ beliefs and help stabilize the economy. Duffy (2006, 2008) gives stimulating surveys of laboratory experiments in various macro settings and how individual and aggregate behavior could be explained by agent-based models.

However, most of these papers cannot be viewed as pure experimental testing of the expectations hypothesis, everything else being constant, because in the experiments dynamic market equilibrium is affected not only by expectations feedback but also by other types of human decisions, such as trading behavior. A number of laboratory experiments have focused on expectation formation exclusively. Schmalensee (1976) presented subjects with historical data on wheat prices and asked them to predict the
mean wheat price for the next five periods. In Dwyer et al. (1993) and Hey (1994),
subjects had to predict a time series generated by a stochastic process such as a random
walk or a simple linear first-order autoregressive process; in the last two papers no
economic context was given. Kelley and Friedman (2002) considered learning in an
orange juice futures price-forecasting experiment, where prices were driven by a lin-
ear stochastic process with two exogenous variables (weather and competing supply).
A drawback common to these papers is that the historical or stochastic price series are
exogenous and there is no feedback from subjects’ forecasting behavior.

4.6.1. Learning to Forecast Experiments

In the remaining part of this section we mainly focus on the learning-to-forecast exper-
iments in Hommes et al. (2005). In these experiments, subjects forecast the price of
a risky asset, which is determined by market clearing with feedback from individ-
ual expectations. Similar experiments have been performed by van de Velden (2001),
Gerber et al. (2002), Sutan and Willinger (2005), Adam (2007), Hommes et al. (2007),
and Heemeijer (2007); see also the recent survey in Duffy (2008). We are particularly
interested in the following questions:

- How do boundedly rational agents form individual expectations, and how do they
  learn in a heterogeneous world?
- How do individual forecasting rules interact, and what is the aggregate outcome of
  these interactions?
- Will coordination occur, even when there is limited market information?
- Does learning enforce convergence to rational expectations equilibrium?

In real financial markets, traders are involved in two related activities: prediction and
trade. Traders make a prediction concerning the future price of an asset, and given this
prediction, they make a trading decision. In the experiments discussed here, the only
task of the subjects is to forecast prices; asset trading is computerized and derived from
optimal demand (from mean-variance maximization), given the individual forecast. The
experiments can therefore be seen as learning to forecast experiments (Marimon and
Sunder 1994, p. 134), in contrast to learning to solve intertemporal optimization prob-
lems or, more concisely, learning to optimize experiments (Duffy 2006, p. 4), where
participants are asked to submit their decisions (e.g., trading or consumption quantities)
while their private beliefs about future developments remain implicit. Learning to fore-
cast experiments provide us with “clean” data on expectations, which can be used to
test various expectations hypotheses.

In the experiments, each participant is told that he or she is an advisor to a pension
fund, with the only task to predict next period’s price of a risky asset. Earnings are given
by a (truncated) quadratic scoring rule:

\[ e_{ht} = \max \left\{ 1300 - \frac{1300}{49} \left( p_t - p_{ht}^e \right)^2, 0 \right\} \]  

(4.63)
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where 1300 points is equivalent to 0.5 euros, and earnings are 0 in period \( t \) when \( |p_t - p_{ht}^e| \geq 7 \). Subjects are informed that their pension fund needs to decide how much to invest in a risk-free asset paying a risk-free gross rate of return \( R = 1 + r \), where \( r \) is the real interest rate, and how much to invest in shares of an infinitely lived risky asset. The risky asset pays uncertain IID dividends \( y_t \) with mean \( \bar{y} \). The mean dividend \( \bar{y} \) and the interest rate \( r \) are common knowledge, so that the subjects could compute the (constant) fundamental \( p^* = \bar{y}/r = 3/0.05 = 60 \). Subjects know that the price of the asset is determined by market clearing. Although they do not know the exact underlying market-clearing equation, they have qualitative information about the market and are informed that the higher their forecast, the larger will be the fraction of money of their pension fund invested in the risky asset and the larger will be the demand for stocks. They do not know the exact investment strategy of their pension fund and the investment strategies of the other pension funds. They also do not know the number of pension funds (which is six) or the identity of the other members of the group.

The experiment lasts for 51 periods. In every period \( t \) the participants have to predict the price \( p_{t+1} \) of the risky asset in period \( t + 1 \), given the available information consisting of past prices \( p_{t-1}, p_{t-2}, \ldots, p_1 \) and the participants’ own past individual predictions \( p_{ht}^e, p_{h,t-2}^e, \ldots, p_{h1}^e \). Notice that the participants have to make a two-period-ahead forecast for \( p_{t+1} \), since \( p_{t-1} \) is the latest available price observation. Subjects are told that their price forecast has to be between 0 and 100 for every period. In periods 1 and 2, no information about past prices is available. At the end of period \( t \), when all predictions for period \( t + 1 \) have been submitted, the participants are informed about the price in period \( t \) and earnings for that period are revealed. On their computer screens, the subjects are informed about their earnings in the previous period, total earnings, a table of the last 20 prices, and their corresponding predictions and time series of the prices and their predictions. Subjects have no information about earnings and predictions of others.

4.6.2. The Price-Generating Mechanism

The asset market is populated by six pension funds and a small fraction of fundamentalist robot traders. Each pension fund \( h \) is matched with a participant and makes an investment decision at time \( t \) based on this participant’s prediction \( p_{h,t+1}^e \) of the asset price. The fundamentalist trader always predicts the fundamental price \( p^f \) and trades based on this prediction.

The realized asset price in the experiment is determined by market clearing, with the pension fund’s asset demand derived from mean-variance maximization, given their advisor’s forecast, as in the standard asset-pricing model with heterogeneous beliefs (e.g., Campbell, Lo, and MacKinlay, 1997; Brock and Hommes, 1998; see Section 4.2). The market-clearing price is given by Eq. 4.5:

\[
\begin{align*}
  p_t &= \frac{1}{1 + r} [(1 - n_t) \bar{p}_t] \\
  \bar{p}_{t+1}^e &= \frac{1}{6} \sum_{h=1}^{6} p_{h,t+1}^e \\
\end{align*}
\]

where \( \bar{p}_{t+1}^e = \frac{1}{6} \sum_{h=1}^{6} p_{h,t+1}^e \) is the average forecast for period \( t + 1 \) of the six participants, \( n_t \) is the time-varying weight of the fundamentalist traders, and \( \epsilon_t \) is a noise term,
representing (small) stochastic demand and supply shocks. Note that the realized asset price $p_t$ at time $t$ is determined by the individual price predictions $p_{h,t+1}$ for time $t+1$. Therefore, when traders have to make a prediction for the price in period $t+1$, they do not know the price in period $t$ yet, and they can only use information on prices up to time $t-1$.

The weight $n_t$ of the fundamental traders in the market is endogenous and depends positively on the absolute distance between the asset price and the fundamental value according to:

$$n_t = 1 - \exp\left(-\frac{1}{200} | p_{t-1} - p^f | \right)$$  \hspace{1cm} (4.65)

The greater this distance, the more the fundamentalist trader will buy or short the asset. The fundamentalist trader therefore acts as a “stabilizing force” pushing prices in the direction of the fundamental price. Their presence excludes the possibility of everlasting speculative bubbles in asset prices.\(^{27}\) Also note that $n_t = 0$, if $p_{t-1} = p^f$.

An important feature of the asset-pricing model is its self-confirming nature or positive feedback: If all traders make a high (low) prediction, the realized price will also be high (low). This feature is characteristic for speculative asset markets: If traders expect a high price, the demand for the risky asset will be high, and as a consequence the realized market price will be high, assuming that the supply is fixed.

### 4.6.3. Benchmark Expectations Rules

Figure 4.13 shows the price dynamics under three benchmark expectation rules: rational expectations, naïve expectations, and a trend extrapolation rule. In the rational expectations’ benchmarks, all agents forecast the price to be equal to its fundamental value $p^f = 60$.\(^{28}\) Realized prices are then given by

$$p_t = p^f + \frac{1}{1 + r} \epsilon_t$$  \hspace{1cm} (4.66)

Therefore, under rational expectations, prices exhibit small random fluctuations around the fundamental price $p^f = 60$. This outcome of the experiment should probably not be expected right from the start, but perhaps subjects can learn to coordinate on the rational, fundamental forecast.

\(^{27}\)DeGrauwe et al. (1993) discuss a similar stabilizing force in an exchange rate model with fundamentalists and chartists. In the same spirit, Kyle and Xiong (2001) introduce a long-term investor who holds a risky asset in an amount proportional to the spread between the asset price and its fundamental value. Since in the experiments the fundamental value is $p^f = 60$, the weight of the fundamentalist traders is bounded above by $\bar{n} = 1 - \exp\left(-\frac{3}{10} \right) \approx 0.26$. The weight of the other traders is the same for each trader and equal to $(1 - n_t)/6 \leq 0.17$.

\(^{28}\)Recall that participants know the values of $\bar{y}$ and $r$ and therefore have enough information to compute the fundamental value and predict it for any period.
Under naïve expectations all participants use the last observed price as their forecast, that is, \( p_{h,t+1}^e = p_{t-1} \). The asset price then converges (almost) monotonically toward the fundamental price, as illustrated in Figure 4.13. Finally, the figure also illustrates what happens when all subjects use the simple trend extrapolation rule:

\[
p_{h,t+1}^e = \frac{(60 + p_{t-1})}{2} + p_{t-1} - p_{t-2}
\]

If all subjects use the forecasting rule (Eq. 4.67), realized market prices will fluctuate for 50 periods. This simple rule may be viewed as an anchor and adjustment heuristic, following the terminology of Tversky and Kahneman (1974), since it uses an anchor (the average of the fundamental price and the last observed price) and extrapolates the last price change from there. One may wonder how subjects would arrive at this anchor.
if they do not know the fundamental, but quite surprisingly a number of subjects used a rule very similar to Eq. 4.67.

4.6.4. Aggregate Behavior

Figure 4.14 shows time series of the realized asset prices and individual predictions in the experiments for five different groups. The first three groups illustrate the three typical qualitatively different outcomes in the treatment with robot traders:

- **Monotonic convergence.** The price converges (atmost) monotonically to the fundamental price from below.
- **Persistent oscillations.** The price oscillates with more or less constant amplitude; there is no convergence of the price to its fundamental value.
- **Dampened oscillations.** The price oscillates around the fundamental price with large amplitude initially, but the amplitude decreases over time, indicating (slow) convergence to the fundamental price.

The last two groups in Figure 4.14 illustrate what happens in a different treatment of the experiments without fundamental robot traders. When there are no fundamental robot traders present in the market, persistent price oscillations with large amplitude typically occur. The difference between these last two groups lies in the upper bound for price predictions, set to 100 (as in the case with robot traders) and 1000, respectively. Hommes et al. (2008) ran experiments without robot traders and a high upper bound of 1000 (maintaining the same fundamental price $p_f = 60$) and in six out of their seven markets long-lasting price bubbles (almost) reaching the upper bound were observed, with price levels up to 15 times the fundamental value.

Comparing the experimental results in Figure 4.14 with the simulated benchmarks in Figure 4.13 one observes that realized prices under naïve expectations resemble realized prices in the case with monotonic convergence remarkably well. On the other hand, the case of persistent oscillatory behavior in the experiment is qualitatively similar to the asset price behavior when participants use a simple $AR(2)$ prediction strategy. Clearly, naïve and $AR(2)$ prediction strategies give a qualitatively much better description of aggregate asset price fluctuations in the experiment than does the benchmark case of rational expectations. Recall from Subsection 4.6.3 that an $AR(2)$ rule has a simple behavioral interpretation as an anchor and adjustment trend-following forecasting strategy.

4.6.5. Individual Prediction Strategies

In this subsection we discuss some characteristics and estimation of individual prediction strategies. Some participants try to extrapolate observed trends and by doing so overreact and predict too high or too low. Other participants are more cautious when submitting predictions and use adaptive expectations, that is, an average of their last forecast and the last observed price. An individual degree of overreaction can be
FIGURE 4.14 Realized prices (a) and individual predictions (b) in five typical asset-pricing experiments. The fundamental price $p^f = 60$ is indicated by a horizontal line. The first three examples show three different outcomes in the experiments with robot traders: monotonic convergence, persistent oscillations, and dampened oscillations. The last two examples show experiments without robot traders for upper bounds of 100 and 1000, respectively. The right panels show a striking coordination of individual forecasts.

quantified as the average absolute (one-period) change in predictions of participant $h$:

$$\Delta^e_h = \frac{1}{41} \sum_{t=11}^{51} |p^\epsilon_h - p^\epsilon_{h,t-1}|$$  \hspace{1cm} (4.68)

The average absolute change in the price is given by $\Delta = \frac{1}{41} \sum_{t=11}^{51} |p_t - p_{t-1}|$. We will say that individual $h$ overreacts if $\Delta^e_h > \Delta$ and we will say that individual $h$ is
FIGURE 4.15 Individual degrees of overreactions for 10 different groups, all with a robot trader: the first seven with a fundamental \( p^f = 60 \) and the last three with a fundamental \( p^f = 40 \). The line segments represent the average absolute price change; the dots represent the average absolute changes in individual forecasts. Dots above the line segments correspond to individual overreaction.

cautious if \( \Delta^c_i \leq \Delta \). Figure 4.15 illustrates the individual degree of overreaction for the different groups. In the case of monotonic convergence (groups 2 and 5), there is no overreaction; in the case of permanent oscillations (groups 1, 6, 8, and 9) a majority of subjects shows some overreaction, but it is relatively small. In the case of dampened oscillations (groups 4, 7, and 10), with large temporary bubbles in the initial phases of the experiment, a majority of participants strongly overreacts. Oscillatory behavior and temporary bubbles are thus caused by overreaction of a majority of agents.

Individual prediction strategies have been estimated using a simple linear model:

\[
p_{h,t+1}^e = \alpha_h + \sum_{i=1}^{4} \beta_{hi} p_{t-i} + \sum_{j=0}^{3} \gamma_{hj} p_{h,t-j} + \nu_t
\]  

(4.69)

where \( \nu_t \) is an IID noise term. This general setup includes several important special cases: (1) naïve expectations (\( \beta_{h1} = 1 \), all other coefficients equal to 0); (2) adaptive expectations (\( \beta_{h1} + \gamma_{h0} = 1 \), all other coefficients equal to 0), and (3) \( AR(L) \) processes (all coefficients equal to 0, except \( a_h, \beta_{h1}, \ldots, \beta_{hL} \)). The estimation results for 60 participants (using observations \( t = 11 \) to \( t = 51 \)) can be summarized as follows:

- For more than 90% of the individuals, the simple linear rule (Eq. 4.69) describes forecasting behavior well.
- In the monotonically converging markets, a majority of subjects uses a naïve, an adaptive, or an \( AR(1) \) forecasting rule.
- In the dampened and persistently oscillating markets, a majority of subjects uses simple \( AR(2) \) or \( AR(3) \) forecasting rules; in particular, a number of subjects use a simple trend-following rule of the form:

\[
p_{h,t+1}^e = p_{t-1} + \delta_h \left( p_{t-1} - p_{t-2} \right) \quad \delta_h > 0
\]  

(4.70)

This forecasting rule corresponds to positive feedback of momentum traders.
Within each group, participants learn to coordinate on a simple forecasting rule, which becomes self-fulfilling. If participants coordinate on an adaptive or AR(1) forecasting rule, the asset price monotonically converges to the fundamental price. In contrast, if the participants coordinate on a trend-following rule, transitory or even permanent price oscillations may arise, with persistent deviations from fundamental price. Anufriev and Hommes (2008) extended the adaptive belief systems in Section 4.2 and developed an evolutionary heuristics-switching model, matching all three different observed patterns in the learning to forecasting experiments remarkably well.

4.6.6. Profitability

In the learning-to-forecast experiments, subjects have been rewarded by their forecasting performance. As discussed in Hommes (2001), the fitness measure of (minus) squared forecasting errors is equivalent to risk-adjusted profits, and therefore it may be a relevant measure in real markets (see Footnote 6). But it is interesting to investigate the corresponding realized profits of the pension funds. In this section therefore we briefly discuss the profitability, that is, the (nonrisk-adjusted) realized profits, of the investment strategies. Realized profits of a mean-variance investment strategy based on a price forecast \( p_{ht+1}^f \) are given by:

\[
\pi_{ht} = (p_{t+1} + y_{t+1} - R p_t)(p_{ht+1}^f + \bar{y} - R p_t)
\]  

(4.71)

As a typical example, Figure 4.16 shows the realized profits and the realized accumulated profits in group 4, that is, a group with a robot trader and upperbound 100 exhibiting dampened price oscillations (the third panel in Figure 4.14). Figure 4.16 shows the realized profits corresponding to the six individual forecasts, the realized profits of the fundamental robot trader, and the realized profits of a hypothetical switching strategy, together with the realized price series (scaled by a factor of 3).

Clearly there are large fluctuations in the realized profits and all strategies occasionally suffer from large losses. The fundamental strategy starts with positive profits in periods 1 to 6 as the asset price rises from below the fundamental. When the asset price rises above its fundamental value and the bubble starts, the fundamental strategy makes large losses in periods 7 to 13. At the peak of the first bubble, at period 13, the fundamental strategy has performed second to last on average, with the second to lowest accumulated realized profits. During the crash, however, the fundamentalists make huge profits because they have built a large short position in the risky asset. At the same time, most other (trend-following) strategies suffer large losses because they hold long positions. As the crash continues and the asset price falls below its fundamental value, the fundamental strategy starts making losses again. At the bottom of the market in period 19, fundamentalists make a large loss. Over the full sample of 50 periods, however, on average the fundamental strategy performs very well and accumulates more profits than the other six forecasting strategies. Figure 4.16 also shows that the fundamental strategy is beaten by a switching strategy, always selecting the best (according to last period’s
FIGURE 4.16 Realized (nonrisk adjusted) profits (top panel) and accumulated profits (bottom panel) for group 4 (see Figure 4.14) with a robot trader and price upper bound 100. The graphs in (a) show periods 1–50; (b) graphs zoom in to periods 1–25; the realized price series (scaled by a factor of 3) is also shown. The top graphs show the realized profits corresponding to the six individual forecasting strategies, the fundamental robot trader, and a hypothetical switching strategy using the best (according to last period’s realized profits) of the other seven strategies. The bottom graphs show the accumulated profits of these eight strategies. Realized profits exhibit large fluctuations over time, and all strategies at times suffer from large losses. On average and in terms of accumulated profits, the fundamental strategy performs very well but is beaten by the switching strategy.

realized profit) out of the seven other strategies. Stated differently, the switching strategy always uses the forecast of the advisor whose pension fund generated the highest realized profit in the previous period. Such a switching strategy beats the fundamental robot strategy.
4.7. CONCLUSION

This chapter has reviewed some behavioral finance models with evolutionary selection of heterogeneous trading strategies and discussed their empirical and experimental validity. When strategy selection is driven by short-run realized profits, trend-following strategies may destabilize asset markets. Asset price fluctuations are characterized by phases in which fundamentalists dominate and prices are close to fundamentals, suddenly interrupted by possibly long-lasting phases of price bubbles when trend-following strategies dominate the market and prices deviate persistently from fundamentals. Even in simple heterogeneous belief models, asset prices are difficult to predict and market timing based on the prediction of the start or the collapse of a bubble is extremely difficult and highly sensitive to noise. Estimation of simple versions of heterogeneous agent models on yearly S&P 500 data suggests that stock prices are characterized by behavioral heterogeneity. Simple evolutionary models therefore could provide an explanation of, for example, the dot-com bubble as being triggered by good news about economic fundamentals and subsequently strongly amplified by trend-following trading strategies. Laboratory experiments using human subjects confirm that coordination on simple trend-following strategies may arise in asset markets and cause persistent deviations from fundamentals.

In a heterogeneous beliefs asset-pricing model, as long as prices fluctuate around their fundamental values, fundamentalist strategies do quite well in terms of accumulated profits. If there are no limits to arbitrage and fundamentalists can survive possibly long-lasting bubbles during which they suffer large losses, their strategy performs very well in the long run and may help stabilize markets. However, fundamentalists can be beaten by a switching strategy based on recent realized profits, thus providing an incentive for investors to keep switching strategies. It should be noted that the models discussed here are very stylized, with a well-defined fundamental price. In real markets, there may be a lot of disagreement about the “correct” fundamental price, and it may then not be so clear what the fundamental strategy would be. Limits to arbitrage may also prevent fundamentalists from holding long-lasting positions opposite to the trend as more and more traders go with the trend based on their recent success.

Most behavioral asset pricing models focus on a single risky asset. Only a few extensions to a multiasset setting have been made until now. Westerhoff (2004) and Chiarella et al. (2007) considered multi-asset markets, where chartists can switch their investments between different markets for risky assets. The interaction between the different markets causes complex asset price dynamics, with different markets exhibiting comovements as well as clustered volatility and fat tails of asset returns. Böhm and Wenzelburger (2005) apply random dynamical systems to investigate the performance of efficient portfolios in a multi-asset market with heterogeneous investors.

Work on complex evolutionary systems in finance is rapidly growing, but little work has been done on policy implications. The most important difference with a representative rational agent framework is probably that, in a heterogeneous, boundedly rational world, asset price fluctuations exhibit excess volatility. If this is indeed the case, it has important implications concerning, for example, the debates on whether a Tobin
tax on financial transactions is desirable or whether financial regulation is desirable. Westerhoff and Dieci (2006) use a complex evolutionary system to investigate the effectiveness of a Tobin tax. Investors can invest in two different speculative asset markets. If a Tobin tax is imposed on one market, it is stabilized, while the other market is destabilized; if a tax is imposed on both markets, price fluctuations in both markets decrease.

Brock, Hommes, and Wagener (2008) study the effects of financial innovation on price volatility and welfare. They extend the asset-pricing model with heterogeneous beliefs in Section 4.2 by introducing hedging instruments in the form of Arrow securities, that is, state contingent claims to uncertain future events. They show that more hedging instruments may destabilize markets and decrease welfare when agents are boundedly rational and choose investment strategies based on reinforcement learning. The intuition of this result is simple: Optimistic and/or pessimistic traders take larger positions when they can hedge more risk, and those who happen to be on the right side of the market will be reinforced more. In a world of bounded rationality and learning from past success, more hedging instruments may thus lead to more persistent deviations from market fundamentals. Developing a theory of complex multi-asset market models with heterogeneous interacting trading strategies and empirical and experimental testing will be an important area of research for years to come. From a practitioner viewpoint, this kind of research seems highly relevant to gain more insight into the causes of financial crises, such as the current credit crisis, in order to hopefully avoid such crises in the future.